Persistent homology - Theory

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Outline

Persistent topology offers a modular, powerful tool for analyzing data of various nature. A very large experimental body in heterogeneous applications is available[1].

[1] Ferri, M., *Persistent topology for natural data analysis - A survey*, In: LNAI, vol 10344, Springer (2017), 117-133

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So far, all applications needed to build a topological space (mostly the space of a simplicial complex) on which to define a filtration and compute homology.

[1] Ferri, M., *Persistent topology for natural data analysis - A survey*, In: LNAI, vol 10344, Springer (2017), 117-133

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In the first two talks I'll go through theory and applications of "classical" persistence. Then I'll illustrate a general definition of *persistence functions*, which allows the use of persistence diagrams without going through simplicial or topological constructions, and the first examples of it.

Recognition (transformation groups)

The simplest type of shape recognition is by superimposition: One tries to deform a template into the given image.

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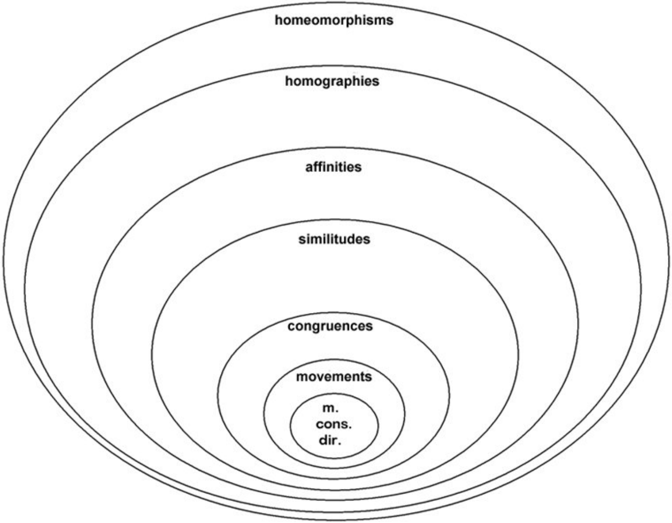
Problem: Different environments imply different transformation groups; the wider the group, the greater is the freedom, but also the computational complexity, due to a greater number of parameters.

Recognition (transformation groups)



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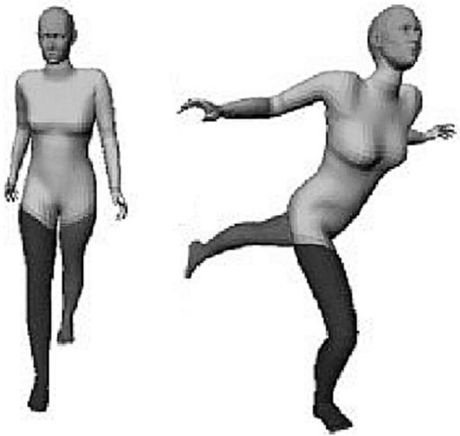
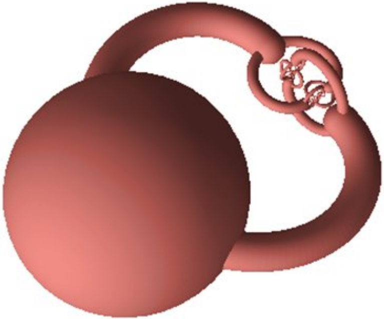


A homeomorphism

Recognition (transformation groups)



But here there is a homeomorphism too!

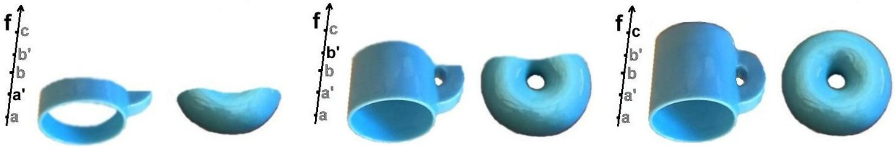
Filtered spaces

In pattern recognition and shape analysis, geometry is too rigid, but topology is too free.
Homeomorphic spaces - like mug and donut - can be very different from an intuitive viewpoint.

Filtered spaces

Things may change by filtering two (even homeomorphic) spaces and studying the topology of the subsets level by level.

In the picture, filtering by sublevel sets of the function “- ordinate” reveals what we might be interested in: The concavity of the mug.



Persistence modules

Definition 3.1 (Sublevel sets)

Given a size pair (X, f), with $f : X \to \mathbb{R}$ continuous, given $u \in \mathbb{R}$, the sublevel set under u is the set $X_u = \{x \in X \mid f(x) \leq u\}$.

Definition 3.2 (Persistence Module [2])

For all $u, v \in \mathbb{R}, \quad u < v$, the inclusion map $\iota^{u,v} : X_u \to X_v$ is continuous and induces, at each degree k, a linear transformation $\iota_*^{u,v} : H_k(X_u) \to H_k(X_v)$.
These linear transformations, with their domains and ranges, build the k-Persistence Module of (X, f).

[2] Zomorodian, A., Carlsson, G., *Computing Persistent Homology*, Discr. Comput. Geom. 33 (2004), 249–274

Persistent Betti Numbers

Definition 3.3 (Persistent Betti Numbers[3])

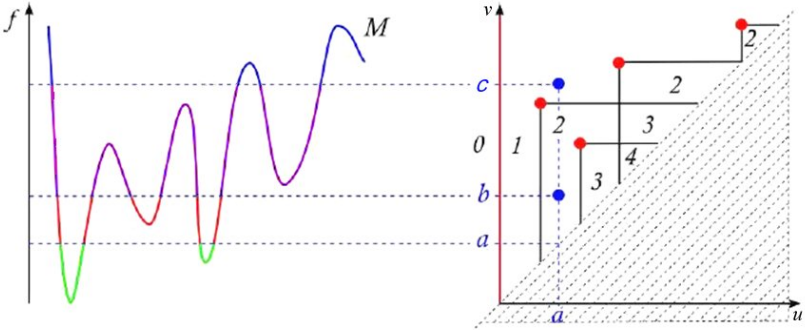
For all $u, v \in \mathbb{R}, \quad u < v$, the *k*-Persistent Betti Number (*k*-PBN) function (also called Rank Invariant) assigns to the pair (u, v) the number

$$\beta_k(u, v) = \dim \operatorname{Im} \iota_*^{u,v},$$

i.e. the number of classes of k-cycles of $H_k(X_u)$ which "survive" in $H_k(X_v)$.

[3] Edelsbrunner, H., Harer, J., *Persistent homology—a survey*. In: Surveys on Discrete and Computational Geometry, Contemp. Math. Amer. Math. Soc., 453 (2008), 257–282

0-PBNs

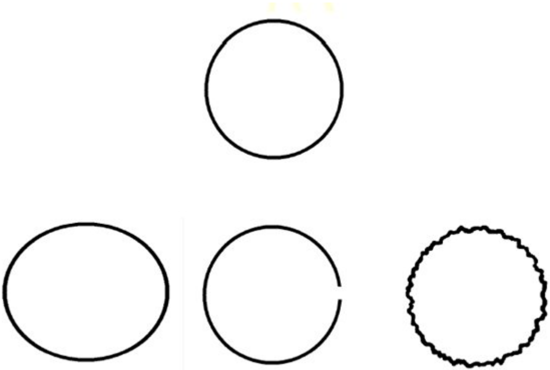


For the pair (M, f) of the picture, the 0-PBN function is shown.

The value 2 found at $(u, v) = (a, b)$ means that, of the three connected components of the sublevel set under b, only two come from under a.

The value 1 found at $(u, v) = (a, c)$ means that the two connected components under a merge into one in the sublevel set under c.

The Bologna – Genova approach: “Size Functions”



Which object has “the same shape” as the upper circle? In our opinion, this depends on the observer (his/her viewpoint, interest, tasks . . .).

The Bologna – Genova approach: “Size Functions”

Teams at Bologna and Genova Universities contrived the 0-PBNs (then called *Size Functions*) for facing various problems of shape classification and pattern analysis.
They used several different filtering functions, in order to capture the viewpoints, the shape ideas of the observer[4][5].

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[4]Frosini, P., *Measuring shapes by size functions*, Proc. of SPIE, Intelligent Robots and Computer Vision X: Algorithms and Techniques, Boston, MA 1607 (1991), 122–133.

[5]Verri, A., Uras, C., Frosini, P., Ferri, M., *On the use of size functions for shape analysis*, Biological Cybernetics 70, (1993), 99–107.

The Colorado – Duke – Stanford approach: topology from sampling

Independently, researchers at Colorado[6], Duke[7] and Stanford[8] Universities devised PBNs for catching the topology of an object through a cloud of sampling points.

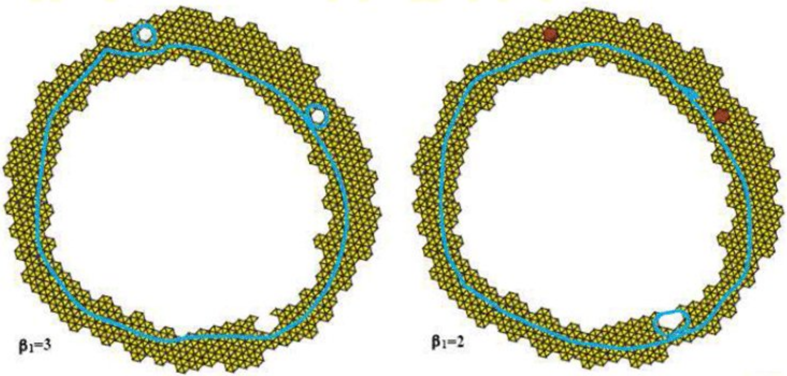
[6] Robins, V., *Computational Topology at Multiple Resolutions*, PhD Thesis, Univ. of Colorado, Boulder (2000)

[7] Edelsbrunner, H., Letcher, D. Zomorodian, A., *Topological persistence and simplification*, Discrete Comput. Geom. 28 (2002), 511–533

[8] Carlsson, G., Zomorodian, A., Collins, A., Guibas, L., *Persistence barcodes for shapes*. In: Proc. Symp. Geom. Process. (2004), 127–138

The Colorado – Duke – Stanford approach: topology from sampling

The filtering function is the radius of balls centered on the sampling points, in a growing Vietoris-Rips complex. The persisting cycles (whence the expression *Persistent Homology*) are likely to be the “true” cycles of the sampled object.



Characterizing properties

Set $\Delta^+ = \{(u, v) \in \mathbb{R} \mid u < v\}$. For the k-PBN $\beta_k^{(X,f)}$ (short: β) of a size pair (X, f) these properties hold[9][10]:

Proposition 5.1

1. *$\beta(u, v)$ is nondecreasing in u and nonincreasing in v;*
2. *for all $u_1, u_2, v_1, v_2 \in \mathbb{R}$ such that $u_1 \leq u_2 < v_1 \leq v_2$ the following inequality holds: $\beta(u_2, v_1) - \beta(u_1, v_1) \geq \beta(u_2, v_2) - \beta(u_1, v_2)$*
3. *given an analogous pair (X', f'), if a homeomorphism $\psi : X \to X'$ exists such that $\sup_{x \in X} |f(x) - f'(\psi(x))| \leq h$ ($h > 0$), then for all $(u, v) \in \Delta^+$ the inequality $\beta_k^{(X,f)}(u - h, v + h) \leq \beta_k^{(X',f')}(u, v)$ holds.*

[9] Frosini, P., Landi, C., *Size functions and formal series*, Appl. Algebra Engrg. Comm. Comput. 12 (2001), 327–349

[10] d'Amico, M., Frosini, P., Landi, C., *Natural pseudo-distance and optimal matching between reduced size functions*, Acta Applicandae Mathematicae 109 (2010), 527–554

A typical shape

Prop. 5.1 (parts 1, 2) implies the typical shape of the k-PBNs: overlapping (possibly unbounded) triangles with horizontal and vertical sides. Then all information can be condensed in some points and lines; they form the k-th Persistence Diagrams (k-PD) of (X, f).

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Given a point $p = (u, v) \in \Delta^+$, let $\varepsilon > 0$ be such that $u + \varepsilon < v - \varepsilon$. Then Prop. 5.1(2) implies that

$\mu_\varepsilon(p) = \beta(u+\varepsilon, v-\varepsilon)-\beta(u-\varepsilon, v-\varepsilon)-\beta(u+\varepsilon, v+\varepsilon)+\beta(u-\varepsilon, v+\varepsilon) \geq 0$

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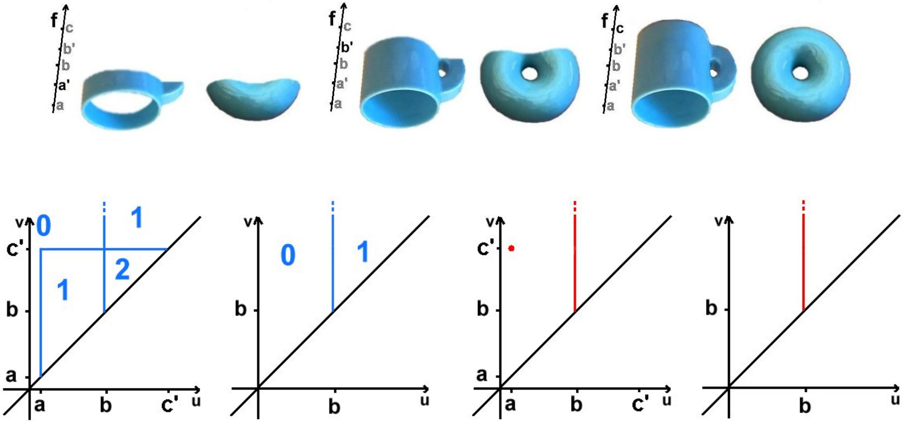
Definition 5.2

A cornerpoint is a point $p \in \Delta^+$ such that $\mu(p) > 0$. Its multiplicity is $\mu(p)$.

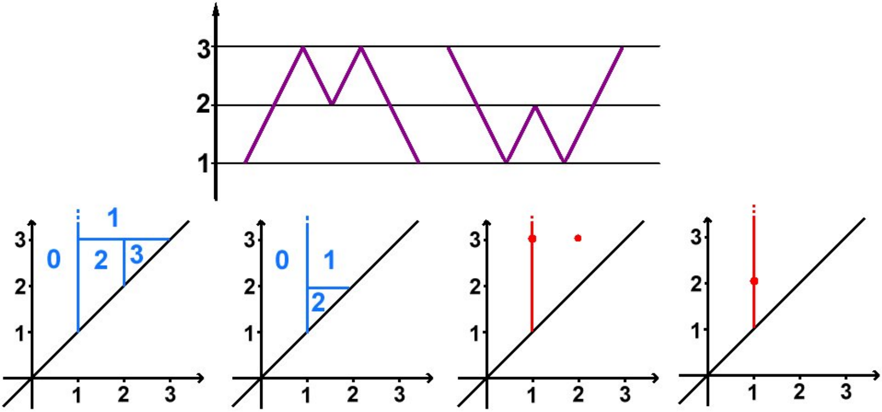
Cornerlines (often replaced by their points at infinity, called cornerpoints at infinity) are defined in a similar way.

A typical shape



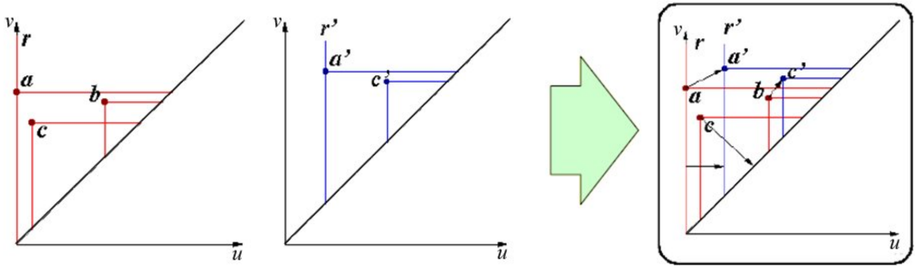
From left to right: 1-PBN functions of mug and of donut, 1-PDs of mug and of donut.

A typical shape



An example of 0-PBNs and of the correspondig 1-PDs.

Bottleneck distance



Definition 5.3

We assume all PDs finite. Given two persistence diagrams D, D' (suitably made equipotent with points on the diagonal), for a fixed bijection between D and D', its *weight* is the max of the L_∞ distance of corresponding points. The bottleneck (or matching) distance $d(D, D')$ is defined as the minimum weight among all bijections.

Invariance

If the filtering function is invariant under a certain group of transformations, the k-PBN functions enjoy the same invariance.

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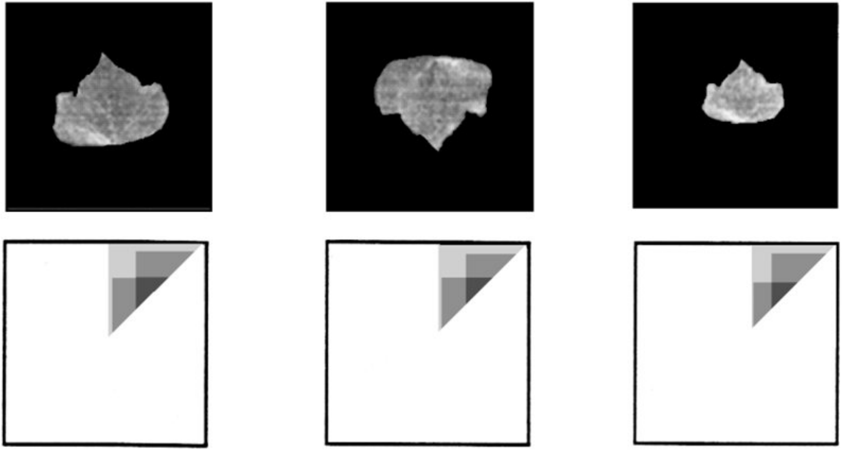
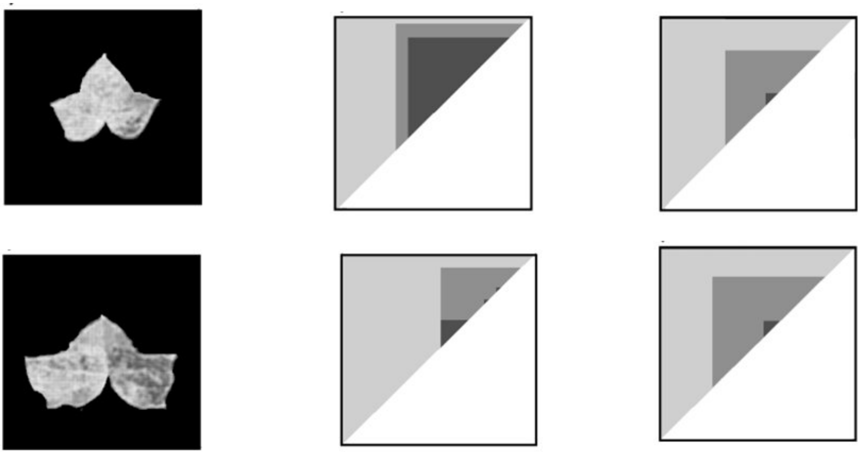
So, e.g., if we want to compare two images which differ by an affinity, we can use an affinity-invariant filtering function and measure the bottleneck distance between the corresponding persistence diagrams. Next there are some examples.

Invariance



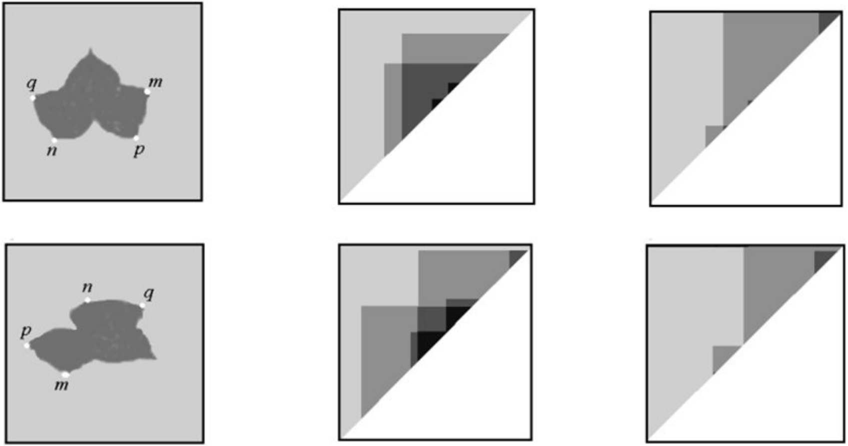
Similitudes

Invariance



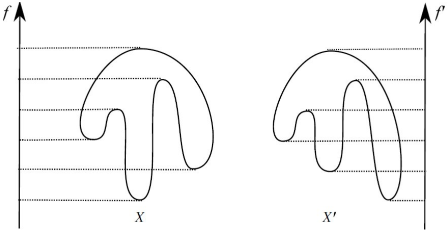
Affinities

Invariance



Homographies

Natural pseudodistance



Definition 5.4

Given two size pairs $(X, f), (X', f')$, with X, X' homeomorphic, their natural pseudodistance is defined as

$$\delta\big((X, f), (X', f')\big) = \inf_\gamma \sup_{x \in X} \|f(x) - f'(\gamma(x))\|_\infty$$

where γ varies in all possible homeomorphisms between X and X'.

Stability and universality

Prop. 5.1 (part 3) implies:

Proposition 5.5 (Stability)

If $D_k(f), D_k(f')$ are the k-PDs of pairs $(X, f), (X, f')$ respectively,

$$d\big(D_k(f), D_k(f')\big) \leq \delta\big((X, f), (X, f')\big)$$

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Moreover,

Proposition 5.6 (Universality)

If $\tilde{d}$ is a distance on persistence diagrams such that, for any $(X, f), (X, f')$

$$\tilde{d}\big(D_k(f), D_k(f')\big) \le \delta\big((X, f), (X, f')\big)$$

then

$$\tilde{d}\big(D_k(f), D_k(f')\big) \le d\big(D_k(f), D_k(f')\big)$$

Easing the classification

Classification of persistence diagrams can be eased by various constructions:

- Landscapes [1,2]
- Gaussian kernels [3,4]
- Persistence paths [5]
- Complex polynomials (see later)
- ...

Moreover, persistence diagrams can be easily fed to a neural network.

Easing the classification

[1] Bubenik, P., *Statistical topological data analysis using persistence landscapes*, The Journal of Machine Learning Research, 16 (2015), 77–102.

[2] Bubenik, P., Dłotko, P., *A persistence landscapes toolbox for topological statistics*, Journal of Symbolic Computation, 78 (2017), 91–114

[3] Donatini, P., Frosini, P., Lovato, A., *Size functions for signature recognition*, Proc. of the SPIE's Workshop "Vision Geometry VII", 3454 (1998), 178–183

4] Ferri, M., Frosini, P., Lovato, A., Zambelli, C., *Point selection: A new comparison scheme for size functions (With an application to monogram recognition)*, Proc. ACCV'98, Springer LNCS 1351 vol. 1 (1998), 329-337

[5] Chevyrev, I., Nanda, V., Oberhauser, H., *Persistence paths and signature features in topological data analysis*, IEEE Transactions on Pattern Analysis and Machine Intelligence (2018), doi: 10.1109/TPAMI.2018.2885516

Variations

Persistent Homology offers quite a lot beyond persistence diagrams:

- Interleaving distance [1,2]
- Zigzag persistence [3]
- Extended persistence [4]
- Filtering functions with circular values [5]
- A_∞-persistence [6]
- Multidimensional filtering functions (see later)
- . . .

Variations

[1] Chazal, F., Cohen-Steiner, D., Glisse, M., Guibas, L.J., Oudot, S.Y., *Proximity of persistence modules and their diagrams*. In: Proc. 25th Annual Symp. on Comput. Geometry, ACM (2009), 237–246

[2] Lesnick, M., *The theory of the interleaving distance on multidimensional persistence modules*, Found. of Comput. Math. 15 (2015), 613–650

[3] Carlsson, G., de Silva, V., Morozov, D., *Zigzag persistent homology and real-valued functions*. In: Proc. 25th Annual Symp. on Comput. Geom., ACM (2009), 247–256

[4] Cohen-Steiner, D., Edelsbrunner, H., Harer, J., *Extending persistence using Poincaré and Lefschetz duality*, Found. of Comput. Math., 9 (2009), 79–103

[5] de Silva, V., Vejdemo-Johansson, M., *Persistent cohomology and circular coordinates*. In: Proc. 25th Annual Symp. on Comput. Geom., ACM (2009), 227–236

[6] Belchí, F., Murillo, A., *A_∞-persistence*. Applicable Algebra in Engineering, Communication and Computing, 26 (2015), 121-139

"Shortened" persistent homology

The bottleneck distance is the best possible. But it is computationally heavy.

[11] Ferri, M., Landi, C., *Representing size functions by complex polynomials*, Proc. Math. Met. in Pattern Recognition 9, Moskow, November 16–19, 1999.

Di Fabio, B., Ferri, M., *Comparing persistence diagrams through complex vectors*, In: ICIAP 2015 Part I; LNCS 9279, Springer (2015), 294-305.

“Shortened” persistent homology

The bottleneck distance is the best possible. But it is computationally heavy.

So we choose the following alternative[11]:

- Warping the plane (due to the special role of the diagonal)
- Considering the transformed PD points as complex numbers
- Forming the complex polynomial having them as roots
- Using only the first few coefficients for comparison.

[11] Ferri, M., Landi, C., *Representing size functions by complex polynomials*, Proc. Math. Met. in Pattern Recognition 9, Moskow, November 16–19, 1999.

Di Fabio, B., Ferri, M., *Comparing persistence diagrams through complex vectors*, In: ICIAP 2015 Part I; LNCS 9279, Springer (2015), 294-305.

Warping

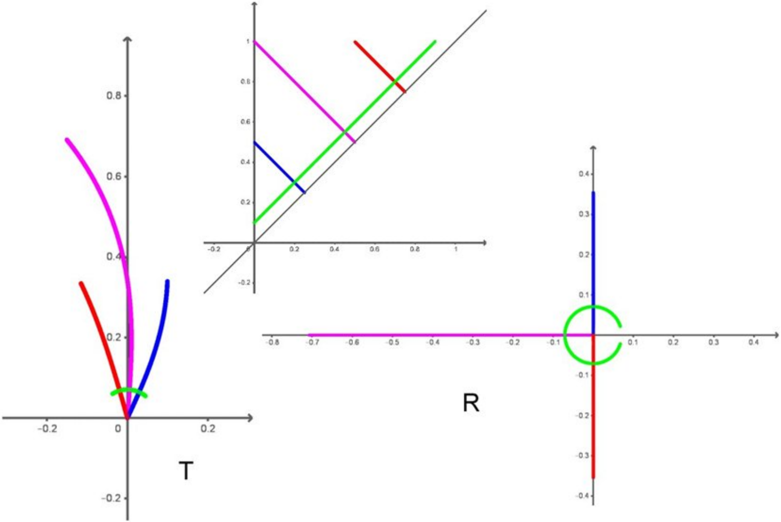
$$T : \bar{\Delta}^+ \to \mathbb{C}, \quad T(u,v) = \tfrac{v-u}{2}(\cos(\alpha) - \sin(\alpha) + i(\cos(\alpha) + \sin(\alpha)))$$

where $\alpha = \sqrt{u^2 + v^2}$.

$$R : \bar{\Delta}^+ \to \mathbb{C}, \quad R(u,v) = \tfrac{v-u}{\sqrt{2}}(\cos(\theta) + i \sin(\theta))$$

where $\theta = \pi(u + v)$.

Warping



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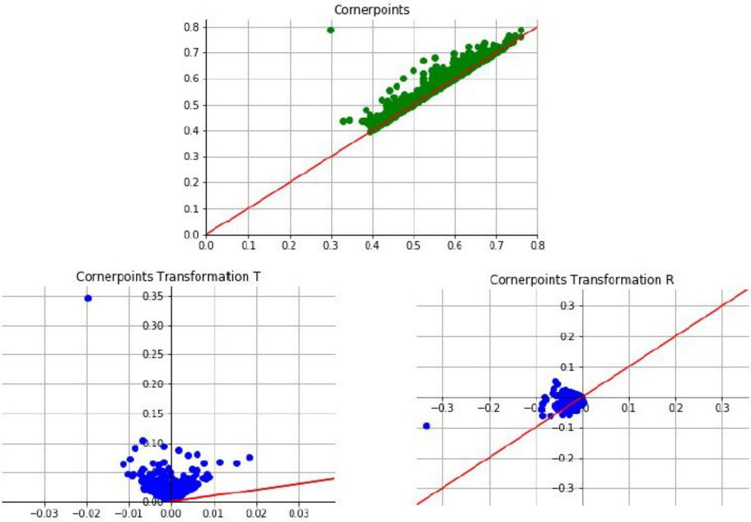
Cornerpoints
Cornerpoints Transformation T
Cornerpoints Transformation R

Viète's formulas

For a polynomial

$$p(t) = b_0 t^n + b_1 t^{n-1} + \cdots + b_{n-1} t + b_n$$

with roots $z_1, \ldots, z_n$

Viètes formulas yield:

$z_1 + z_2 + \cdots + z_n = -\frac{b_1}{b_0}$
$(z_1 z_2 + \cdots + z_1 z_n) + (z_2 z_3 + \cdots + z_2 z_n) + \cdots + (z_{n-1} z_n) = \frac{b_2}{b_0}$
$(z_1 z_2 \cdots z_n) = (-1)^n \frac{b_n}{b_0}$

Shortening the vectors

For each warped persistence diagram we choose the monic polynomial ($b_0 = 1$) with those roots and form the vector

$$\left(-b_1, \sqrt{b_2}, \sqrt[3]{-b_3}, \ldots, \sqrt[N]{(-1)^N b_N}\right)/N$$

where N is the number of cornerpoints of the diagram.

For classification and retrieval we shorten the vector to the first k elements (usually $k = 5, 10, 20, 50$).
Finally, we compare by L^1 distance.

Extending the range

- We define the following relation $\preceq$ ($\prec$) in $\mathbb{R}^n$: if $\vec{u} = (u_1, \ldots, u_n)$ and $\vec{v} = (v_1, \ldots, v_n)$, we write $\vec{u} \preceq \vec{v}$ ($\vec{u} \prec \vec{v}$) if and only if $u_j \leq v_j$ ($u_j < v_j$) for $j = 1, \ldots, n$. Let also Δ^+ be now the set $\{(\vec{u}, \vec{v}) \in \mathbb{R}^n \times \mathbb{R}^n \,|\, \vec{u} \prec \vec{v}\}$.

[12] Frosini, P., Mulazzani, M., *Size homotopy groups for computation of natural size distances*, Bull. of the Belgian Math. Soc., 6 (1999), 455–464

Carlsson, G., Zomorodian, A., *The theory of multidimensional persistence*, Discr. Comput. Geometry, 42 (2009), 71–93

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- We denote by $X\langle \vec{f} \preceq \vec{u} \rangle$ the sublevel set $\{p \in X \mid \overrightarrow{f}(p) \preceq \overrightarrow{u}\}$.

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- We denote by $X\langle\vec{f} \preceq \vec{u}\rangle$ the sublevel set $\{p \in X \mid \overrightarrow{f}(p) \preceq \overrightarrow{u}\}$.
- Persistent Betti Number functions can then be defined in the same way on Δ^+ [12].

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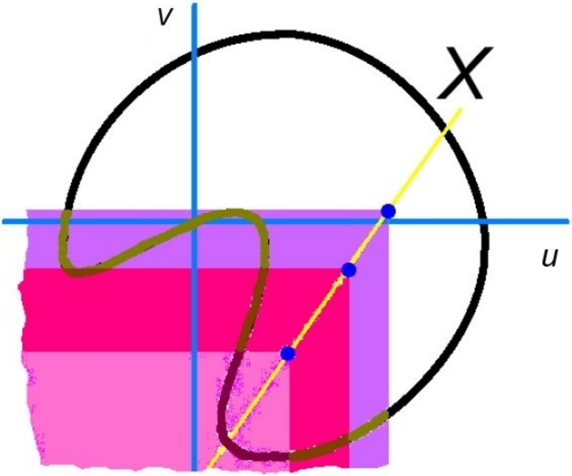
Extending the range



The need for a multidimensional range

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Filtering functions φ with $\mathbb{R}^n$, $n > 1$, as a range, arise quite naturally in applications (e.g. coordinates, RGB, curvature and torsion, . . .).

In general, persistent topology of the single components of φ carries less information than the whole function.

The next pictures illustrate the case of $(\mathcal{C}, \vec{\varphi}')$, $(\mathcal{S}, \vec{\varphi}'')$, where $\mathcal{C}$ and $\mathcal{S}$ are a cube of edge length 2 and a sphere of diameter 2 respectively, and $\vec{\varphi}'(x, y, z) = \vec{\varphi}''(x, y, z) = (|x|, |y|)$.

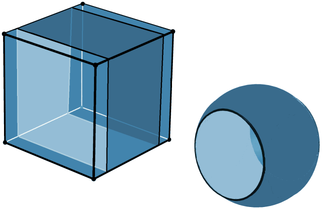
The need for a multidimensional range

Filtering functions φ with $\mathbb{R}^n$, $n > 1$, as a range, arise quite naturally in applications (e.g. coordinates, RGB, curvature and torsion, ...).

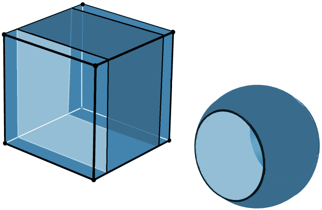
In general, persistent topology of the single components of φ carries less information than the whole function.

The next pictures illustrate the case of $(\mathcal{C}, \vec{\varphi}')$, $(\mathcal{S}, \vec{\varphi}'')$, where $\mathcal{C}$ and $\mathcal{S}$ are a cube of edge length 2 and a sphere of diameter 2 respectively, and $\vec{\varphi}'(x, y, z) = \vec{\varphi}''(x, y, z) = (|x|, |y|)$.

Sublevel sets of the single components are homotopically circles in both cases, whereas they differ if the whole functions are taken into account.



Sublevel sets of the component $|y| \leq \frac{\sqrt{2}}{2}$. The ones of $|x|$ are just rotated versions.



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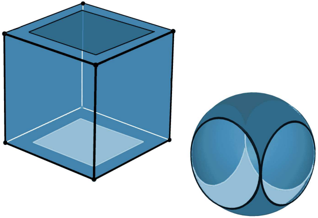

Sublevel sets of $\vec{\varphi}' = \vec{\varphi}'' \preceq (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$.

The problem of discontinuities

Recall that, when $n = 1$, discontinuities occur along line segments, and are determined by the set of cornerpoints, i.e. by a submanifold of dimension 0 of the 2-dimensional Δ^+

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The problem of discontinuities

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In general, in the $2n$-dimensional Δ^+, the rôle of cornerpoints is played by a $(2n - 2)$-dimensional submanifold.

No more possibility of representing the invariant by a formal series?
No more bottleneck distance?

Admissible pairs

For 0-PBNs [13] then generally for k-PBNs [14] we have proved that suitable foliations of Δ^+ exist, along which the computation can be reduced to the one-dimensional case, so back to cornerpoints.

[13] Silvia Biasotti, Andrea Cerri, Patrizio Frosini, Daniela Giorgi, Claudia Landi, *Multidimensional size functions for shape comparison*, J. Math. Imaging and Vision, vol. 32 (2008), n. 2, 161–179

[14] Cagliari, F., Di Fabio, B., Ferri, M., *One-dimensional reduction of multidimensional persistent homology*, Proc. Amer. Math. Soc. 138 (2010), 3003–3017

Admissible pairs

For every vector $\vec{l} = (l_1, \ldots, l_n)$ in $\mathbb{R}^n$ such that $\left\|\vec{l}\right\|_2 = 1$, and $l_j > 0$ for $j = 1, \ldots, n$, and for every vector $\vec{b} = (b_1, \ldots, b_n)$ in $\mathbb{R}^n$ such that $\sum_{j=1}^{n} b_j = 0$, we shall say that the pair $(\vec{l}, \vec{b})$ is *admissible*. Given an admissible pair $(\vec{l}, \vec{b})$, we define the half-plane $\pi_{(\vec{l},\vec{b})}$ in $\mathbb{R}^n \times \mathbb{R}^n$:

$$\begin{cases} \vec{u} & = & s\vec{l} + \vec{b} \\ \vec{v} & = & t\vec{l} + \vec{b} \end{cases}$$

for $s, t \in \mathbb{R}$, with $s < t$.

(The use of the 1-norm for $\vec{l}$ instead of the 2-norm is equivalent)

Tame functions

Each admissible pair identifies a line in $\mathbb{R}^n$. The corresponding half-planes $\pi_{(\vec{l},\vec{b})}$ foliate $\mathbb{R}^n \times \mathbb{R}^n$.

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A continuous function $f : X \to \mathbb{R}$ is tame if it has a finite number of homological critical values and the homology modules of all sublevel sets are finite-dimensional for all $i \in \mathbb{Z}$.

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Let $(X, \vec{\varphi})$ be a size pair. We shall say that $\vec{\varphi}$ is max-tame if, for every admissible pair $(\vec{l}, \vec{b})$, the function $g(P) = \max_{j=1,\dots,n} \left\{ \frac{\varphi_j(P)-b_j}{l_j} \right\}$ is tame.

1D reduction

Theorem 7.1

Let $(\vec{l}, \vec{b})$ be an admissible pair and $\vec{\varphi} = (\varphi_1, \ldots, \varphi_n) : X \to \mathbb{R}^n$ be a max-tame function. Then, for every $(\vec{u}, \vec{v}) = (s\vec{l} + \vec{b}, t\vec{l} + \vec{b}) \in \pi_{(\vec{l},\vec{b})}$, and for

$$g(P) = \max_{j=1,\ldots,n} \left\{ \frac{\varphi_j(P) - b_j}{l_j} \right\}$$

the equality

$$\beta_k^{(X,\vec{\varphi})}(\vec{u}, \vec{v}) = \beta_k^{(X,g)}(s, t)$$

holds for all $i \in \mathbb{Z}$ and $s, t \in \mathbb{R}$ with $s < t$.

Bottleneck distance (multidimensional case)

The previous theorem gives us the opportunity of defining a distance between n-dimensional PBNs.

Definition 7.2

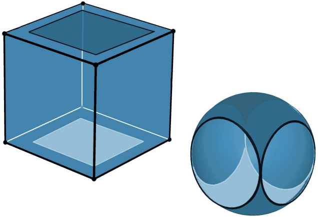
Let $(X, \vec{\varphi}')$, $(Y, \vec{\varphi}'')$ be two size pairs and $\bar{\beta}_k^{(X,\vec{\varphi}')}$, $\bar{\beta}_k^{(Y,\vec{\varphi}'')}$ be the respective PBN functions. Then the k-th bottleneck distance is

$$D(\bar{\beta}_k^{(X,\vec{\varphi}')}, \bar{\beta}_k^{(Y,\vec{\varphi}'')}) = \sup_{(\vec{l},\vec{b})} \min_{j=1,\dots,n} l_j \cdot \ d(\beta_k^{(X,\vec{\varphi}')}, \beta_k^{(Y,\vec{\varphi}'')})$$

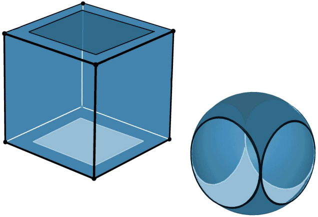
where $(\vec{l}, \vec{b})$ varies among all admissible pairs, and $\beta_k^{(X,\vec{\varphi}')}$, $\beta_k^{(Y,\vec{\varphi}'')}$ are the PBNs of the corresponding one-dimensional functions.

Stability and **universality** hold also in this case, but admissible pairs and max-tameness are necessary ingredients.

Example



Example



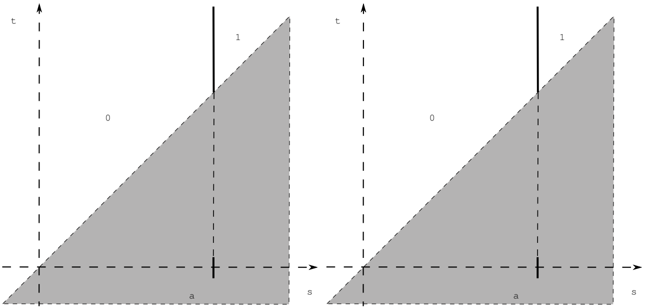
We choose $\vec{l} = (\cos\theta, \sin\theta)$ with $0 < \theta < \frac{\pi}{2}$, and $\vec{b} = (a, -a)$ with $a \in \mathbb{R}$. The corresponding half-plane is parameterized as

$$\begin{cases} u_1 = s\cos\theta + a \\ u_2 = s\sin\theta - a \\ v_1 = t\cos\theta + a \\ v_2 = t\sin\theta - a \end{cases}$$

with $s, t \in \mathbb{R}, s < t$. In particular, we consider $\theta = \frac{\pi}{4}, a = 0$.

Example

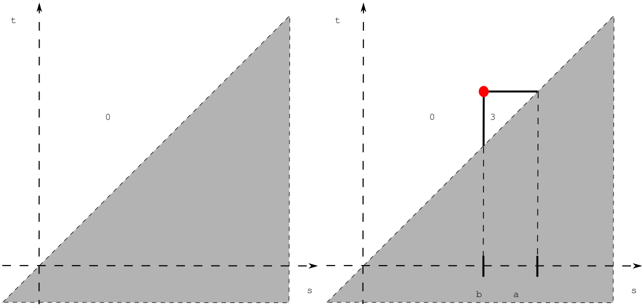
$k = 2$ Left: cube; right: sphere.



$$\Rightarrow D(\overline{\rho}'_{\mathcal{C},\in}, \overline{\rho}''_{\mathcal{S},\in}) \geq \frac{\sqrt{2}}{2} d(\rho'_{\mathcal{C},\in}\ \rho''_{\mathcal{S},\in}) = 0$$

Example

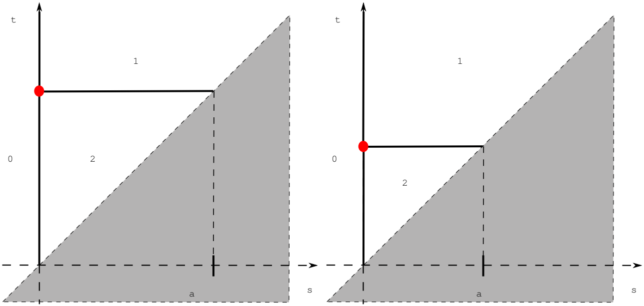
$k = 1$ Left: cube; right: sphere.



$$\Rightarrow D(\overline{\rho}'_{\mathcal{C},\infty}, \overline{\rho}''_{\mathcal{S},\infty}) \geq \frac{\sqrt{2}}{2} d(\rho'_{\mathcal{C},\infty}\ \rho''_{\mathcal{S},\infty}) = \frac{\sqrt{2}}{2}\left(\frac{\sqrt{2}-1}{2}\right)$$

Example

$k = 0$ Left: cube; right: sphere.



$$\Rightarrow D(\overline{\rho}'_{\mathcal{C},\prime}, \overline{\rho}''_{\mathcal{S},\prime}) \geq \frac{\sqrt{2}}{2} d(\rho'_{\mathcal{C},\prime}, \rho''_{\mathcal{S},\prime}) = \frac{\sqrt{2}}{2}(\sqrt{2} - 1)$$

The 2D case

In the 2-dimensional case, admissible pairs are simply of the the type $(\vec{l},\vec{b}) = \big((a, 1-a), ((b, -b)\big)$, so the 1-dimensional reduction leads us to study the persistence diagrams of

$$f_{(a,b)}(p) := \max\left\{\frac{f_1(p)-b}{a}, \frac{f_2(p)+b}{1-a}\right\}$$

In particular: is there a preferred slope for comparing (i.e. distinguishing) two shapes?

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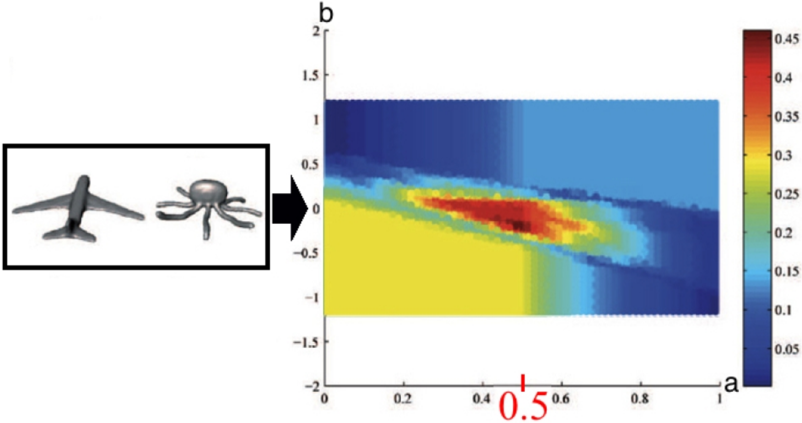
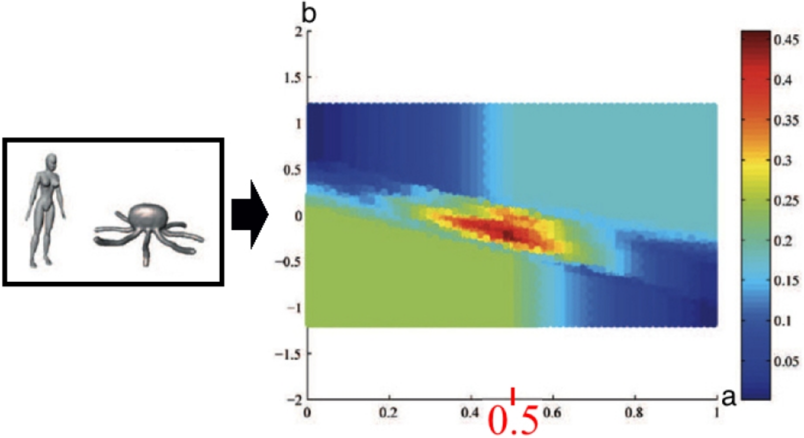
In particular: is there a preferred slope for comparing (i.e. distinguishing) two shapes?

Here are some pictures illustrating some results that we have obtained by means of the 2D bottleneck distance. The objects that we compare are displayed on the left of each figure. The color at the point (a, b) represents the value of the distance of the corresponding 1D reduced persistence diagrams. The largest values are in **red** and **brown**, the lowest ones are in **blue**.

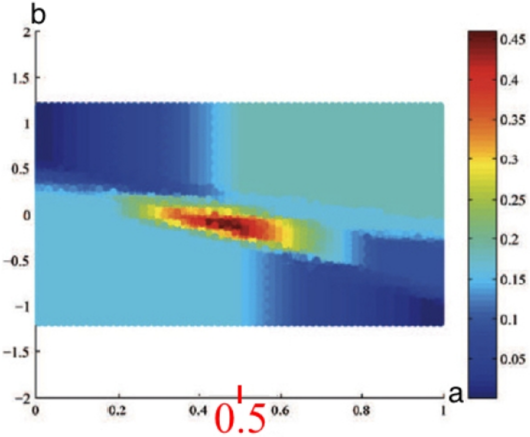
The 2D case


b
2
1.5
1
0.5
0
−0.5
−1
−1.5
−2
0
0.2
0.4
0.5
0.6
0.8
1
a
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05

The 2D case


b
2
1.5
1
0.5
0
−0.5
−1
−1.5
−2
0
0.2
0.4
0.5
0.6
0.8
1
a
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05

The 2D case


b
2
1.5
1
0.5
0
−0.5
−1
−1.5
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0.2
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0.5
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0.8
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a
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0.1
0.05

The 2D case

Intuitive as it may appear, the conjecture that $a = 1/2$, i.e. slope =1, is the best possible choice for 1D reduction in comparing shape, has not yet been proved.

[15] Cerri, A., Ethier, M., Frosini, P., *A study of monodromy in the computation of multidimensional persistence*, Proc. 17th IAPR Int. Conf. on Discrete Geometry for Computer Imagery, LNCS 7749, Springer (2013), 192–202

The 2D case

Intuitive as it may appear, the conjecture that $a = 1/2$, i.e. slope =1, is the best possible choice for 1D reduction in comparing shape, has not yet been proved.

But in the study of the conjecture a very interesting phenomenon emerged: monodromy [15].

[15] Cerri, A., Ethier, M., Frosini, P., *A study of monodromy in the computation of multidimensional persistence*, Proc. 17th IAPR Int. Conf. on Discrete Geometry for Computer Imagery, LNCS 7749, Springer (2013), 192–202

Monodromy

There may be a pair $\big((\bar{a}, 1-\bar{a}), (\bar{b}, -\bar{b})\big)$ for which a cornerpoint of multiplicity ≥ 2 occurs. If we move to a nearby pair $\big((a', 1-a'), (b', -b')\big)$ the multiple cornerpoint doubles into two simple ones.

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What happens if we circle around $\big((\bar{a}, 1-\bar{a}), (\bar{b}, -\bar{b})\big)$ in the plane ab, starting from $\big((a', 1-a'), (b', -b')\big)$ and coming back to the same pair (i.e. to the same leaf of the foliation of $\mathbb{R}^2 \times \mathbb{R}^2$)?

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What happens if we circle around $\big((\bar{a}, 1-\bar{a}), (\bar{b}, -\bar{b})\big)$ in the plane *ab*, starting from $\big((a', 1-a'), (b', -b')\big)$ and coming back to the same pair (i.e. to the same leaf of the foliation of $\mathbb{R}^2 \times \mathbb{R}^2$)?

Of course the two simple cornerpoints move around and come back to the same starting position, but INVERTED!

`./monodromia/monodromy.mov`

We are studying this phenomenon from various viewpoints.

END OF THE FIRST PART

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lorenzo.zuffi@studio.unibo.it

All our software concerning graph persistence can be found at
`https://gitlab.com/mattia.bergomi/perscomb`

Persistent homology - Applications

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[1] Dip. di Matematica and ARCES, Univ. di Bologna, Italia (Vision Mathematics group)

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Fundação Champalimaud, Lisboa, Portugal

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Dip. di Matematica, Univ. di Bologna, Italia

Outline

1. **Introduction**
2. **Classification**
3. **Retrieval**

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2. Classification
3. Retrieval

Modularity and flexibility represent a relevant edge of persistent homology with respect to other shape descriptors: whatever the input pair (X, f), the shape of the outcoming peristence diagram is standard. This means that you can build an effective classifier of PDs and you get a classifer of a wide variety of object types.

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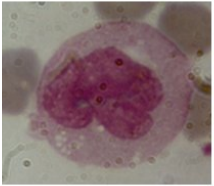
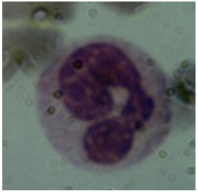
Add to this the fact that (algebraic) topology captures — by its very nature — quality in a quantitative way, and you get a great tool for classification and retrieval of 2D, 3D, 4D shapes, above all of a natural origin, but also of more elusive “shapes”. Here are some examples, ranging from the pioneering works of the ’90s up to recent research.

1. Introduction
2. **Classification**
3. Retrieval

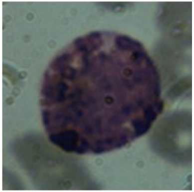
Leukocytes



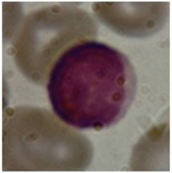
Neutrophile granulocyte



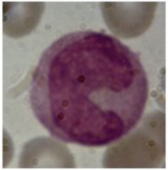
Eosinophile granulocyte



Basophile granulocyte



Lymphocyte



Monocyte

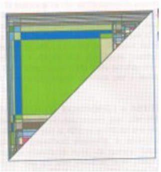
Leukocytes

- Goal[1]: classification (up to confusion eosinophile-neutrophile)
- The space: boundary of starlike hull of cell
- Filtering functions: along radii from centre of mass
  - Sum of grey tones
  - Max variation
  - Sum of pixel-pixel variations (in absolute value)

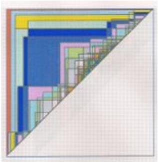
[1] Ferri, M., Lombardini, S., Pallotti, C., *Leukocyte classification by size functions*, Proc. 2nd IEEE Workshop on Applications of Computer Vision, Sarasota, 1994 Dec. 5-7 (1994), 223–229

Leukocytes

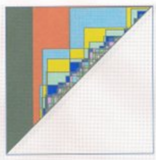
Non-basophile granulocyte



Basophile granulocyte



Lymphocyte



Monocyte

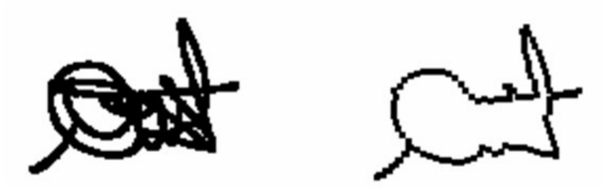
Filtering function: sum of grey tones along radii

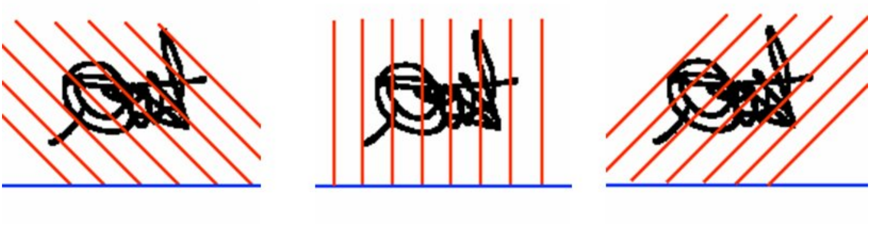
Monograms

- Goal[2]: personal identification
- Space1: outline of monogram
- Filtering function:
  - Distance from centre of mass
- Space2: a segment (horizontal)
- Filtering functions:
  - Number of black pixels along segments (3 directions)
  - Number of pixel-pixel black-white jumps (3 directions)

[2] Ferri, M., Frosini, P., Lovato, A., Zambelli, C., *Point selection: A new comparison scheme for size functions (With an application to monogram recognition)*, Proc. ACCV'98, Hong Kong 8–10 Jan. 1998, Springer LNCS 1351 vol. 1 (1998), 329–337

Monograms



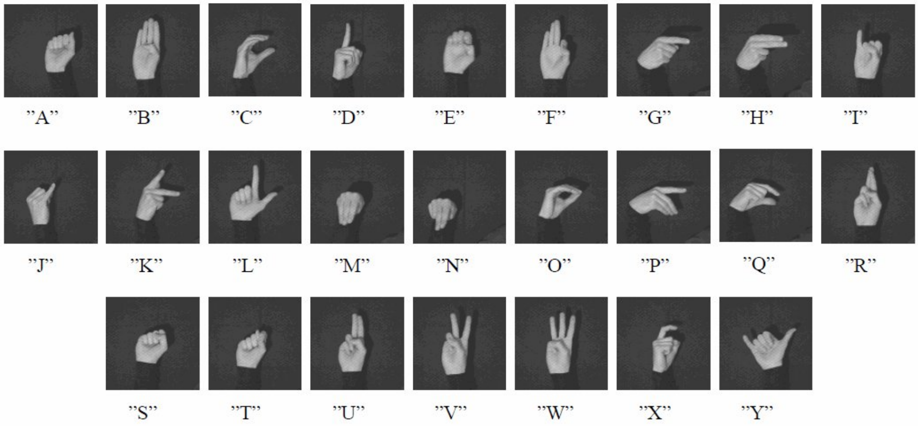


Monograms

- Fuzzy characteristic functions obtained from normalized inverse of distance
- Weighted average of characteristic functions

Live demo performed at ACCV'98

Sign alphabet



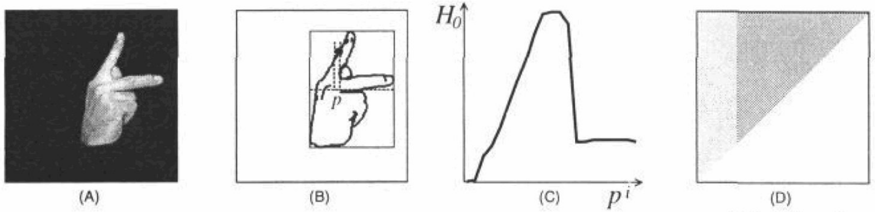
Sign alphabet

A. Verri, Genova (Italy)

- Goal[3]: recognition. Signs performed with glove on uniform background
- The space: horizontal baseline segment
- Filtering functions: for each point, maximum distance of a contour point within a strip of fixed width, with 24 different strip orientations

[3] Uras, C., Verri, A., *On the Recognition of the Alphabet of the Sign Language through Size Functions*, Proc. XII Int. Conf. IAPR, Jerusalem (1994), 334 – 338

Sign alphabet

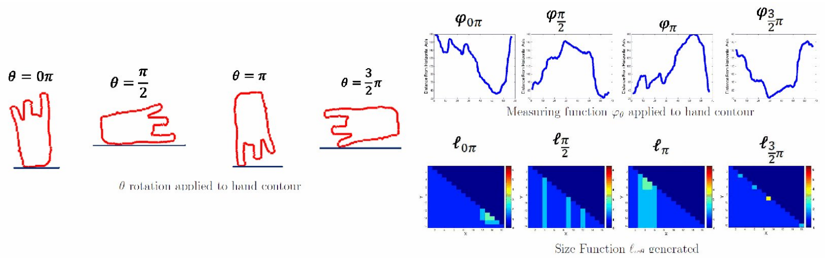


Live demo performed at the 12th IAPR Conference

Sign alphabet

D. Kelly (Maynooth, Ireland)

- Goal [4]
- The space: contour
- Filtering functions: distance from four lines

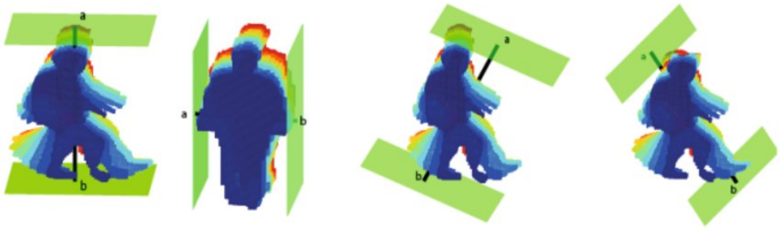


[4] Kelly, D., McDonald, J., Lysaght, T., Markham, Ch., *Analysis of Sign Language Gestures Using Size Functions and Principal Component Analysis*, Proc. IMVIP2008, Portrush, Northern Ireland (2008), 31–36

Human gait

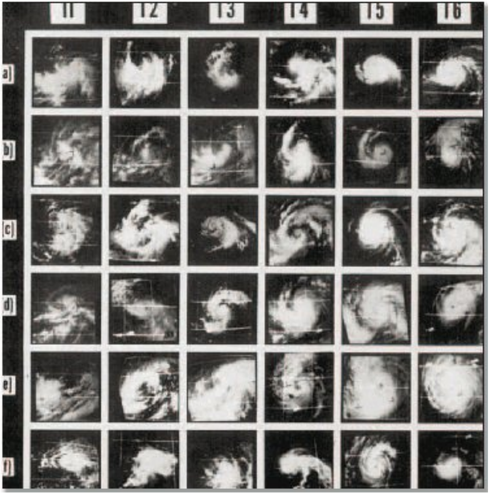
J. Lamar-León, Habana (Cuba), R. Gonzalez-Diaz, Sevilla (Spain)

- Goal[5]: personal identification and surveillance
- The space: 3D stack of silhouettes
- Filtering functions: distance from fixed planes



[5] Lamar-León, J., García-Reyes, E.B., Gonzalez-Diaz, R., *Human gait identification using persistent homology*. In: CIARP 2012, LNCS 7441, Springer (2012), 244–251

Tropical cyclones



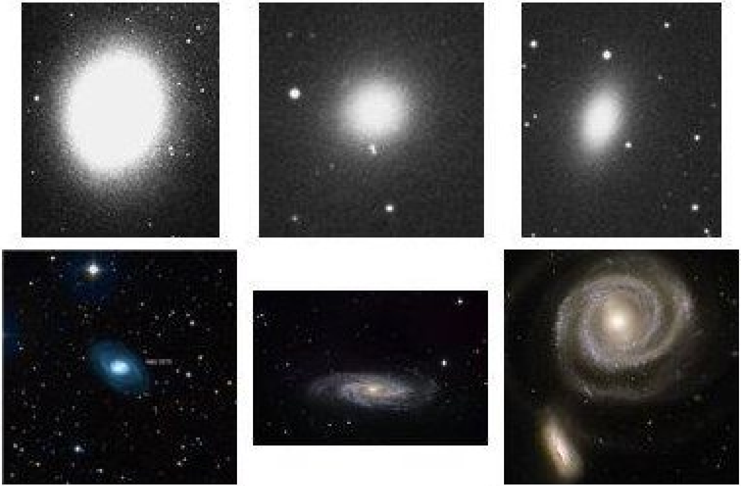
Tropical cyclones

S. Banerjee, Kolkata (India)

- Goal[6]: evaluate risk and intensity of forming hurricane
- The space: time interval
- Filtering functions: two characteristic measures of cyclones:
  - Central Feature portion
  - Outer Banding Feature

[6]Banerjee, S., *Size Functions in the Study of the Evolution of Cyclones*, Int. J. Meteorology 36(358) (2011), 39–46

Galaxies



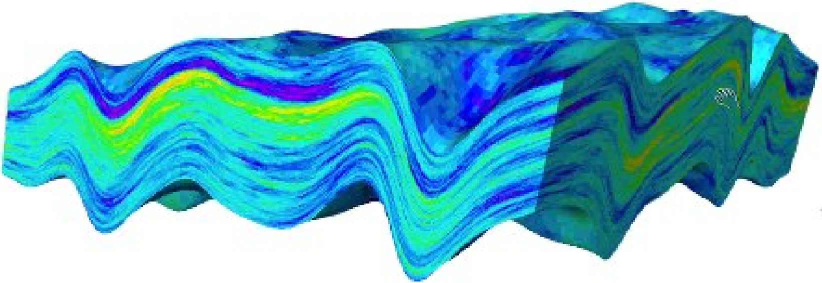
Galaxies

S. Banerjee, Kolkata (India)

- Goal[7]: classification
- The space: the image rectangle
- Filtering functions:
  - ratio between major and minor axis of isophotes
  - Ringermacher-Mead pitch
  - colour-based parameter

[7] Banerjee, S., *Size functions in galaxy morphology classification*, Int. J. Comput. Appl. 100 (2014), 1–4

Oil and gas reservoirs



Realization of an oil reservoir by SGS

Oil and gas reservoirs

V.A. Baikov, Ufa; R.R. Gilmanov, A.A. Yakovlev, Sankt-Petersburg; Ya.V. Bazaikin, I.A. Taimanov, Novosibirsk (Russia)

- Goal [8]: estimate differences between reservoirs and between digital models of reservoirs
- Space: cubical cell model
- Filtering function: permeability

[8] Baikov, V. A., Gilmanov, R. R., Taimanov, I. A., Yakovlev, A. A., *Topological characteristics of oil and gas reservoirs and their applications*, In: Towards Integrative Machine Learning and Knowledge Extraction, LNAI 10344, Springer (2017), 182–193

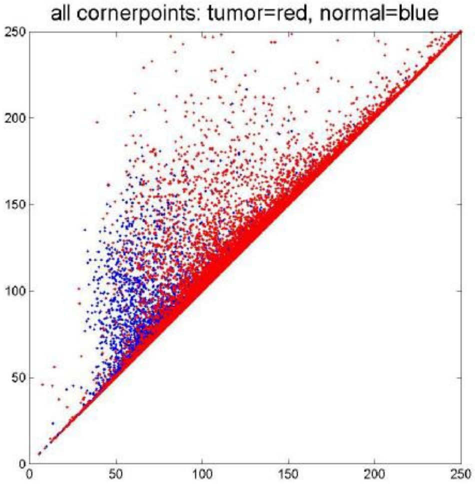
Mouth cells

A. Micheletti, Milano (Italy); G. Landini, London (UK)

- Goal[9]: tumor diagnosis
- Space: the image rectangle
- Filtering function: distance from centre of mass

[9] Micheletti, A., Landini, G., *Size functions applied to the statistical shape analysis and classification of tumor cells*, In: Proc. ECMI 2006, Mathematics in Industry 12, Springer (2008), 538–542

Mouth cells



Melanocytic lesions

Melanocytic lesion images acquired under polarized light and mild magnification

Melanoma Naevus

Problems:

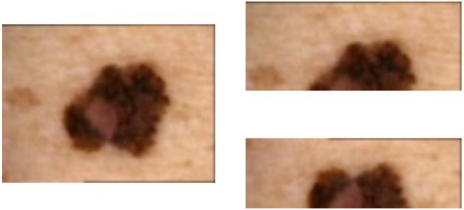
- No template for either class
- Various diagnostic criteria
- Morphological analysis not always sufficient

Melanocytic lesions

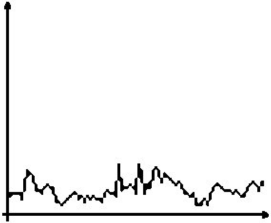
- Goal[10]: risk assessment.
- We took a bundle of 45 lines through the center of mass, and for each we compared the two halves of the lesion, separated by the line by computing the distance of 0-PBN's of the two halves. The resulting functions provided the parameters we used.
- Space: splitting line
- Filtering functions:
    - distance from the splitting line
    - sum of luminance along perpendicular segments
    - sum of color variations along perpendicular segments

[10] Ferri, M., Stanganelli, I., *Size functions for the morphological analysis of melanocytic lesions*, Int. J. Biomed. Imaging 2010 (2010), Article ID 621357, doi:10.1155/2010/621357

Melanocytic lesions



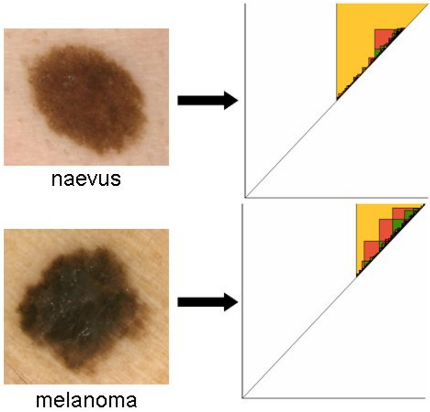
An image and one of
its splittings



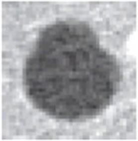
Distance as a function
of the splitting line
(filt. fct.: luminance)

Melanocytic lesions

We fed an SVM with parameters extracted from these functions and from a bumpiness measure coming from the 0-PBNs of filtering function distance from centre of mass:

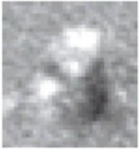


Hepatic lesions

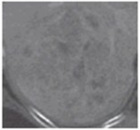


(a) Cyst

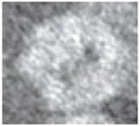
(b) Metastasis



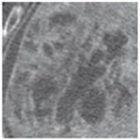
(c) Hemangioma



(d) HCC



(e) Focal Nodule



(f) Abscess

(g) NeN

(h) Laceration

(i) Fat Deposit

Hepatic lesions

G. Carlsson, Stanford (USA)

- Goals[11]: classification of lesions, comparison of 1D and 2D persistence
- Space: the image rectangle
- Filtering function: the pair (grey tone, distance from cell boundary)

[11] Adcock, A., Rubin, D., Carlsson, G., *Classification of hepatic lesions using the matching metric*, Comput. Vis. Image Underst. 121 (2014), 36–42

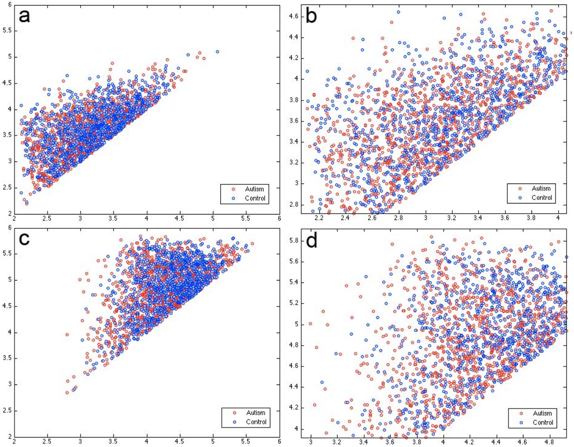
Brain cortex

M.K. Chung, Madison; P. Bubenik, Cleveland (USA); P.T. King, Guelph (Canada)

- Goal[12]: finding cues of autism
- Space: cortical mesh obtained from MRI
- Filtering function: cortical thickness

[12]Chung, M.K., Bubenik, P., Kim, P.T., *Persistence diagrams of cortical surface data*, In: Proc. IPMI 2009, LNCS 5636, Springer (2009), 386–397

Brain cortex



a (magnified in b): 0-PD; c (magnified in d): 1-PD. red: autism, blue: normal

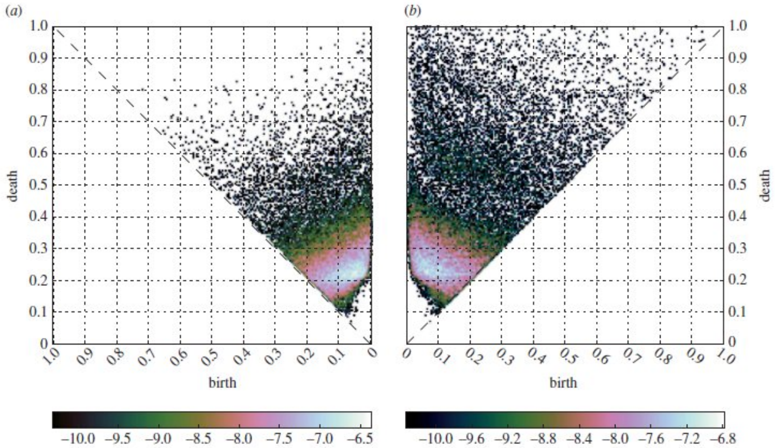
Brain connections

L.D. Lord, Oxford; F. Turkheimer, London (UK); F. Vaccarino, Torino (Italy)

- Goal[13]: understanding brain connection modification under the assumption of psilocybine
- Space: simplicial complex of cliques built on the graph of connections
- Filtering function: - functional connectivity

[13] Lord, L.-D., Expert, P., Fernandes, H.M., Petri, G., Van Hartevelt, T.J., Vaccarino, F., Deco, G., Turkheimer, F., Kringelbach, M.L., *Insights into brain architectures from the homological scaffolds of functional connectivity networks*, Front. Syst. Neurosci. 10, 85 (2016)

Brain connections



Probability densities for H_1 generators: placebo (left) and psilocybin (right) treated

Brain connections

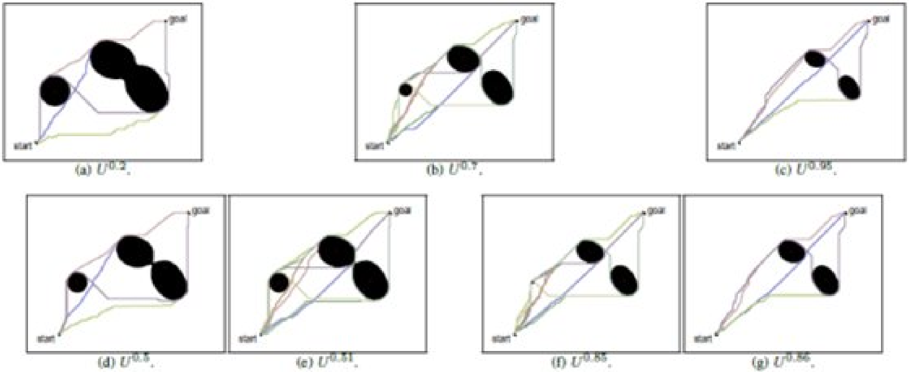
Several other teams have studied the brain structure through persistent homology of simplicial complexes built on a graph:

Giusti, C., Pastalkova, E.,Curto,C., Itskov,V., *Clique topology reveals intrinsic geometric structure in neural correlations*, Proc. Natl. Acad. Sci. 112 (2015), 13455–13460

Reimann, M.W., Nolte, M., Scolamiero, M., Turner, K., Perin, R., Chindemi, G., Dłotko, P., Levi, R., Hess, K. and Markram, H., *Cliques of neurons bound into cavities provide a missing link between structure and function*, Frontiers in Computational Neuroscience, 11 (2017), 48

Sizemore, A. E., Giusti, C., Kahn, A., Vettel, J. M., Betzel, R. F., Bassett, D. S., *Cliques and cavities in the human connectome*, J. Comp. Neuroscience, 44 (2018), 115–145

Robot navigation



(a) $U^{0.2}$. (b) $U^{0.7}$. (c) $U^{0.95}$.

(d) $U^{0.5}$. (e) $U^{0.51}$. (f) $U^{0.85}$. (g) $U^{0.86}$.

Robot navigation

R. Ghrist, Philadelphia (USA)

- Goal[14]: planning robust path for autonomous vehicles
- Space: the space of all paths from start to goal
- Filtering function: probability of occupancy

[14] Bhattacharya, S., Ghrist, R., Kumar, V., *Persistent homology for path planning in uncertain environments*, IEEE Transactions on Robotics, 31 (2015), 578–590

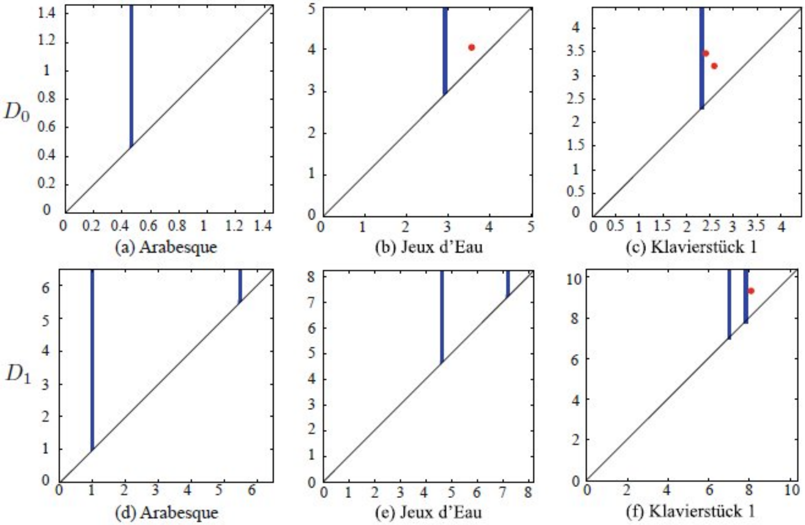
Music

M.G. Bergomi, Lisboa (Portugal)

- Goal[15]: distinguishing musical genres
- Space: a modified Euler's Tonnetz
- Filtering function: note total duration

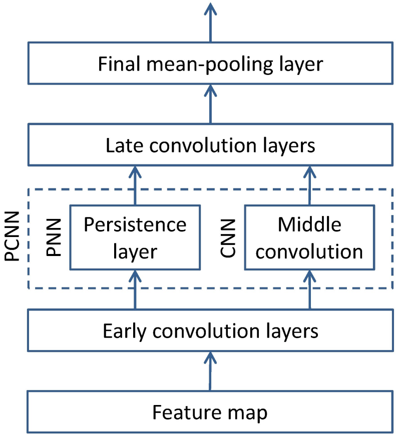
[15] Bergomi, M.G., Baratè, A., Di Fabio, B., *Towards a topological fingerprint of music*, In: Proc. CTIC 2016. LNCS, vol. 9667, Springer (2016), 88–100

Music


D_0
D_1
(a) Arabesque
(b) Jeux d'Eau
(c) Klavierstück 1
(d) Arabesque
(e) Jeux d'Eau
(f) Klavierstück 1

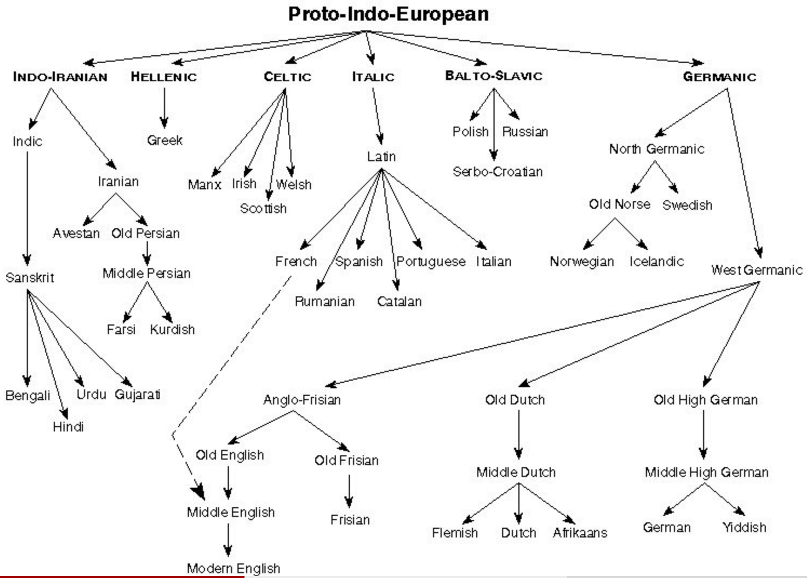
Music

Among various other teams studying music through persistence, of particular interest is the work of the team of J.-Y. Liu, Taipei (Taiwan), which uses persistence landscapes integrated in a convolutional neural network.



Liu, J.-Y., Jeng, S.-K., Yang, Y.-H., *Applying topological persistence in convolutional neural network for music audio signals*, arXiv preprint arXiv:1608.07373 (2016)

Languages


Proto-Indo-European
INDO-IRANIAN
HELLENIC
CELTIC
ITALIC
BALTO-SLAVIC
GERMANIC
Indic
Iranian
Greek
Manx
Irish
Welsh
Scottish
Latin
Polish
Russian
Serbo-Croatian
North Germanic
Old Norse
Swedish
Avestan
Old Persian
Middle Persian
Farsi
Kurdish
Sanskrit
Bengali
Urdu
Gujarati
Hindi
French
Spanish
Portuguese
Italian
Rumanian
Catalan
Norwegian
Icelandic
West Germanic
Anglo-Frisian
Old Dutch
Old High German
Old English
Old Frisian
Middle English
Frisian
Modern English
Middle Dutch
Flemish
Dutch
Afrikaans
Middle High German
German
Yiddish

Languages

M. Marcolli, Pasadena (USA)

- Goal[16]: understanding the structure of language networks
- Space: Vietoris-Rips complex on the languages as points in a Euclidean parameter space
- Filtering function: distance

[16] Port, A., Gheorghita, I., Guth, D., Clark, J. M., Liang, C., Dasu, S., Marcolli, M., *Persistent topology of syntax*, Mathematics in Computer Science, 12 (2018), 33-50

Collaboration networks

S. Pal, Cambridge (USA)

- Goal[17]: studying the difference between collaboration networks through their temporal evolution
- Space: the simplicial complex of cliques on the collaboration graph
- Filtering function: time

[17] Pal, S., Moore, T. J., Ramanathan, R., Swami, A. , *Comparative topological signatures of growing collaboration networks* In: Proc. Int. Workshop on Complex Networks, Springer (2017), 201–209

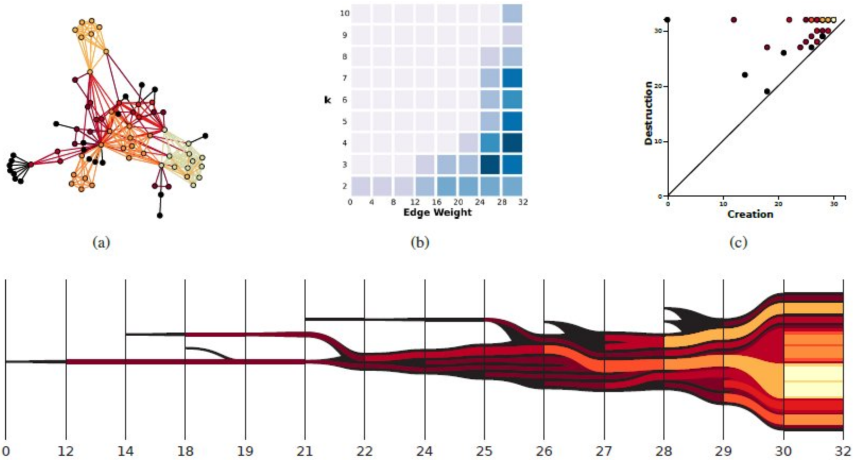
Complex networks

B. Rieck, Kaiserslautern (Germany)

- Goal[18]: analysis and comparison of networks
- Space: the community clique simplicial complex built on the graph
- Filtering function: interaction intensity

[18] Rieck, B., Fugacci, U., Lukasczyk, J., Leitte, H., *Clique community persistence: A topological visual analysis approach for complex networks* IEEE Transactions on Visualization and Computer Graphics, 24 (2017), 822–831

Complex networks



(a) (b) (c)

(d)

I shall show only two experiences of our team, one very old and one recent inspired by the previous one.

The environment is the one of search engines which try to adapt to the user taste and intent, in the context of Content Based Image Retrieval.

Trittico

The user was supposed to provide three examples (as different as possible) of the concept he/she is searching for.

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A set of filtering functions yield corresponding 0-PDs for the three objects.

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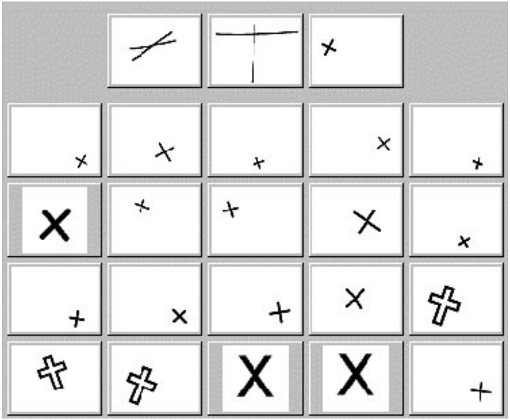
A set of filtering functions yield corresponding 0-PDs for the three objects.

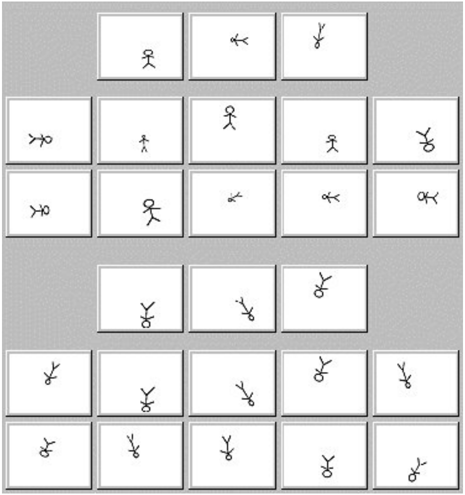
The filtering functions whose PDs are most similar for the three objects are then enhanced in a composite distance.

Trittico



Note the appearance of crosses which are quite different from the input ones

Trittico



The system adapts to an implicit transformation group

Relevance feedback

A first application of Trittico's ideas was in a work using directly the bottleneck distance between PDs[19]

[19]Giorgi, D., Frosini, P., Spagnuolo, M., Falcidieno, B., *3D relevance feedback via multilevel relevance judgements*, The Visual Computer, 26 (2010),1321–1338

[20]Angeli, A., Ferri, M., Monti, E., Tomba, I., *Shortened persistent homology for a biomedical retrieval system with relevance feedback*, In: "Machine Learning and Knowledge Extraction, CD-MAKE 2018", LNCS 11015, Springer (2018), 282–292

Relevance feedback

A first application of Trittico's ideas was in a work using directly the bottleneck distance between PDs[19]

We have then modified both the distance and the criterion[20]. The distance is between the strings of the first few coefficients of the complex polynomials associated to the 0-PDs. The experimentation was performed on a set of melanocytic lesions.

[19] Giorgi, D., Frosini, P., Spagnuolo, M., Falcidieno, B., *3D relevance feedback via multilevel relevance judgements*, The Visual Computer, 26 (2010),1321–1338

[20] Angeli, A., Ferri, M., Monti, E., Tomba, I., *Shortened persistent homology for a biomedical retrieval system with relevance feedback*, In: "Machine Learning and Knowledge Extraction, CD-MAKE 2018", LNCS 11015, Springer (2018), 282–292

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$$D^{AVG} = \frac{d^{(1)} + \cdots + d^{(J)}}{J}$$

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$$D^{AVG} = \frac{d^{(1)} + \cdots + d^{(J)}}{J}$$

With this distance we extract a first output of L images $x_1, \ldots, x_L$ from the dataset.

Relevance feedback

We require the operator to provide his/her vector of (pseudo)distances $\delta = (\delta_1, \ldots, \delta_l)$ of the L output images $x_1, \ldots, x_L$ from the query q.

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We then form the matrix $\mathbf{d}$ whose element d_s^r is $d^{(s)}(q, x_r)$ and look, by a least square method, for nonnegative coefficients $\lambda = (\lambda_1, \ldots, \lambda_J)$ minimizing $\|\mathbf{d}\lambda - \delta\|_2^2$.

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Out of them we form the new distance

$$D^{OUT} = \sum_{j=1}^{J} \lambda_j d^{(j)}$$

END OF THE SECOND PART

mattia.bergomi@neuro.fchampalimaud.org
massimo.ferri@unibo.it
alessandro.mella3@unibo.it
antonella.tavaglione@studio.unibo.it
pietro.vertechi@neuro.fchampalimaud.org
lorenzo.zuffi@studio.unibo.it

All our software concerning graph persistence can be found at
`https://gitlab.com/mattia.bergomi/perscomb`

Non-topological persistence - Theory

Massimo Ferri[1]

[1] Dip. di Matematica and ARCES, Univ. di Bologna, Italia (Vision Mathematics group)

Work performed in collaboration with

Mattia G. Bergomi, Pietro Vertechi

Fundação Champalimaud, Lisboa, Portugal

Alessandro Mella, Antonella Tavaglione, Lorenzo Zuffi

Dip. di Matematica, Univ. di Bologna, Italia

Outline

Non(-necessarily)-topological persistence

We have seen that quite a lot of very different classification and analysis problems can be faced by passing through the persistence paradigm: once you get to associating persistence diagrams to the objects of interest, then you have a whole lot of possibilities for condensing information, for making vectors of them, for measuring distances etc.

Non(-necessarily)-topological persistence

So far, the way to take this train was mandatory: you had to organize your data in the form of a “size pair” (X, f), where X is either a topological space or a simplicial complex, and f has X as a domain and *mathbbR* as a range. The only freedom was in the choice of an alternative range.

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In the frequent case that data occur as a graph, the topology of the graph as a simplicial complex is generally not meaningful. So an auxiliary construction turned out to be necessary: the complex of cliques, the complex of neighborhoods, the complex of clique communities were the most common.

Non(-necessarily)-topological persistence

All things considered, what one actually needs is that data lead to persistence diagrams. Is it possible to get PDs without necessary going through topological spaces or simplicial complexes?

[1] Bergomi, M.G., Ferri, M., Vertechi, P., Zuffi, L., *Beyond topological persistence: Starting from networks* (2019) preprint available at https://arxiv.org/abs/1901.08051

Non(-necessarily)-topological persistence

All things considered, what one actually needs is that data lead to persistence diagrams. Is it possible to get PDs without necessary going through topological spaces or simplicial complexes?

Our idea[1] is to require as axioms those properties which logically imply the structure of the persistent Betti number functions. This will be the definition of Persistence Functions.

[1] Bergomi, M.G., Ferri, M., Vertechi, P., Zuffi, L., *Beyond topological persistence: Starting from networks* (2019) preprint available at https://arxiv.org/abs/1901.08051

Non(-necessarily)-topological persistence

But it's not enough to give an abstract notion, however general; that's why we also give two ways of producing persistence functions: Coherent samplings and steady or ranging properties.

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First it is necessary to specify the context in which this is possible: A particular, but wide type of category.

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Examples are given with (non-topological, non-simplicial) structures in graphs.

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Examples are given with (non-topological, non-simplicial) structures in graphs.

A further, categorical extension is finally given.

Characterizing properties

Let's see what we should generalize and what the widest possible context could be. I now recall some properties of Persistent Betti numbers (PBNs). Out of 1 and 2 all the structure of Persistence Diagrams (PDs) arises; 3 grants stability.

Set $\Delta^+ = \{(u, v) \in \mathbb{R} \mid u < v\}$. For the k-PBN $\beta_k^{(X,f)}$ (short: β) of a size pair (X, f) we have:

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Set $\Delta^+ = \{(u, v) \in \mathbb{R} \mid u < v\}$. For the k-PBN $\beta_k^{(X,f)}$ (short: β) of a size pair (X, f) we have:

Proposition 2.1

1. *$\beta(u, v)$ is nondecreasing in u and nonincreasing in v;*
2. *for all $u_1, u_2, v_1, v_2 \in \mathbb{R}$ such that $u_1 \leq u_2 < v_1 \leq v_2$ the following inequality holds: $\beta(u_2, v_1) - \beta(u_1, v_1) \geq \beta(u_2, v_2) - \beta(u_1, v_2)$*
3. *given an analogous pair (X', f'), if a homeomorphism $\psi : X \to X'$ exists such that $\sup_{x \in X} |f(x) - f'(\psi(x))| \leq h$ ($h > 0$), then for all $(u, v) \in \Delta^+$ the inequality $\beta(u - h, v + h) \leq \beta(u, v)$ holds.*

Concrete categories

We want to generalize 1, 2 and possibly 3 to a suitable category **C**. For generalizing 3, we just need to substitute “homeomorphism” with “**C**-isomorphism”. The problem is upstream: We need the possibility of a filtration of a **C**-object in sublevel objects through a function.

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A *concrete* category is a pair $(\mathbf{C}, \mathcal{U})$ where **C** is a category and $\mathcal{U}$ is a faithful functor $\mathcal{U} : \mathbf{C} \to \mathbf{Set}$.

Categories with canonical subobjects

The basic idea was: In a concrete category $(\mathbf{C}, \mathcal{U})$ we pick up a $\mathbf{C}$-object X and define the filtering function on the corresponding set $\mathcal{U}(X)$, then filter X through the filtration of the set.

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The basic idea was: In a concrete category $(\mathbf{C}, \mathcal{U})$ we pick up a $\mathbf{C}$-object X and define the filtering function on the corresponding set $\mathcal{U}(X)$, then filter X through the filtration of the set.

The problem is that determining subobjects of X through subsets of $\mathcal{U}(X)$ is absolutely non trivial. What is tricky, is the notion of subobject. We need a canonical one.

In this context,for a subset $Z \overset{\iota}{\hookrightarrow} \mathcal{U}(X)$, we can consider subobjects $S \overset{\phi}{\hookrightarrow} X$ such that $\mathcal{U}(\phi)(\mathcal{U}(S)) \subseteq Z$. They form a full subcategory $\{S \overset{\phi}{\hookrightarrow} X \mid \mathcal{U}(\phi)(\mathcal{U}(S)) \subseteq Z\} \subseteq \mathbf{C}_X$ that we will denote $\mathbf{C}_X\restriction_Z$.

Categories with canonical subobjects

Definition 2.2

We say that a concrete category $(\mathbf{C}, \mathcal{U})$ has *canonical subobjects* if the following three conditions are verified:

1. $\mathbf{C}$ has pullbacks and the functor $\mathcal{U}$ preserves pullbacks.
2. for every object $X \in \mathbf{C}$ and for every subset $Z \subseteq \mathcal{U}(X)$, if there is a subobject $T \overset{\chi}{\hookrightarrow} X$ such that $Z = \mathcal{U}(\chi)(\mathcal{U}(T))$ then the category $\mathbf{C}_X\restriction_Z$ has a terminal object $U \overset{\upsilon}{\hookrightarrow} X$. We call such $U \overset{\upsilon}{\hookrightarrow} X$ a *canonical subobject* associated to Z, denoted by $\mathcal{U}^{-1}(Z)$.
3. every morphism $Y \xrightarrow{\chi} X$ can be factored as $Y \xrightarrow{\phi} W \overset{\psi}{\hookrightarrow} X$ where ψ is a monomorphism and $\mathcal{U}(\psi)(\mathcal{U}(W)) = \mathcal{U}(\chi)(\mathcal{U}(Y))$, or equivalently $\mathcal{U}(\phi)$ is surjective. If ψ is canonical, we will call the pair of morphisms $Y \xrightarrow{\phi} W \overset{\psi}{\hookrightarrow} X$ a *canonical factorization* of χ.

Categories with canonical subobjects

As already mentioned, the preceding definition applies to a lot of usual "working" categories: the ones of groups, rings, modules, simplicial complexes, topological spaces, . . .

The definitions which follow can be given in any category with canonical subobjects. Still, for the sake of clearness, **subsequent definitions and examples will be given in the category of undirected graphs**, which has canonical objects.

Graphs as complexes

Graphs will be simple throughout, and will be thought of as 1D simplicial complexes. Homomorphisms, and in particular isomorphisms, will be simplicial maps.

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Given a weighted graph (G, f), where $G = (V, E)$ and $f : E \to \mathbb{R}$ is a *filtering function*, one can extend f to a filtering function $\bar{f} : V \cup E \to \mathbb{R} \cup \{\infty\}$ by defining it as ∞ on isolated vertices and on any other vertex v as the minimum value of f on its incident edges.

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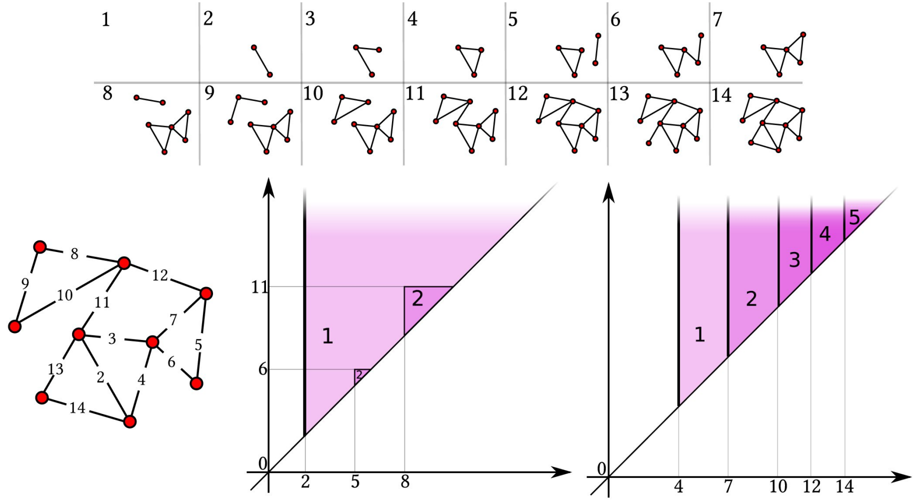
Therefore, a weighted graph gives rise to persistent Betti number functions and persistence diagrams in a natural way.

Graphs as complexes



Natural pseudodistance

Let now $(G,f),(G',f')$, with $G=(V,E)$, $G'=(V',E')$ be weighted graphs and H be the (possibly empty) set of isomorphisms from G to G'.

Definition 2.3

The *natural pseudodistance* of (G,f) and (G',f') is

$$\delta\big((G,f),(G',f')\big)=\begin{cases}\infty & \text{if}\quad H=\emptyset\\ \inf_{\phi\in H}\sup_{e\in E}\left|f(e)-f'\big(\phi(e)\big)\right| & \text{otherwise}\end{cases}$$

Persistence functions

Recall that $\Delta^+ = \{(u, v) \in \mathbb{R} \,|\, u < v\}$, $\Delta = \{(u, v) \in \mathbb{R} \,|\, u = v\}$ and $\overline{\Delta}^+ = \Delta^+ \cup \Delta$.

Let (G, f) be any weighted graph. For each $t \in \mathbb{R}$, the *sublevel graph* G_t is the subgraph of G induced by $f^{-1}\big((-\infty, t]\big)$.

Assume we have a function Λ_G defined on all inclusions between subgraphs of G, with values in the nonnegative integers, and such that $\Lambda_G(\iota) = 0$ if ι has the empty set as domain. Define $\lambda_{(G,f)}(u, v) = \Lambda_G(\iota)$, where ι is the inclusion of G_u into G_v.

Persistence functions

Definition 3.1 (Persistence function)

All functions $\lambda_{(G,f)} : \Delta^+ \to \mathbb{Z}$ are said to be *persistence functions* if conditions 1 and 2 are satisfied; they are said to be *stable* persistence functions if also 3 holds:

1. $\lambda_{(G,f)}(u, v)$ is nondecreasing in u and nonincreasing in v;
2. for all $u_1, u_2, v_1, v_2 \in \mathbb{R}$ such that $u_1 \leq u_2 < v_1 \leq v_2$ the following inequality holds:
$\lambda_{(G,f)}(u_2, v_1) - \lambda_{(G,f)}(u_1, v_1) \geq \lambda_{(G,f)}(u_2, v_2) - \lambda_{(G,f)}(u_1, v_2)$
3. given an analogous pair (G', f'), if an isomorphism $\psi : G \to G'$ exists such that $\sup_{e \in E} |f(e) - f'(\psi(e))| \leq h$ $(h > 0)$, then for all $(u, v) \in \Delta^+$ the inequality $\lambda_{(G,f)}(u - h, v + h) \leq \lambda_{(G',f')}(u, v)$ holds.

Stability

Remark 3.2

A set of theorems holds, granting that any persistence function (conditions 1 and 2) $\lambda_{(G,f)}$ has the same structure as Persistent Betti Numbers functions.

In particular, it can be summarized by a *persistence diagram* $D(f)$ with the usual cornerpoints (proper and at infinity).
$d\big(D(f), D(f')\big)$ will be the usual bottleneck distance.

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In particular, it can be summarized by a *persistence diagram* $D(f)$ with the usual cornerpoints (proper and at infinity).
$d\big(D(f), D(f')\big)$ will be the usual bottleneck distance.

Theorem 3.3 (Stability)

For weighted graphs $(G, f), (G', f')$ as above, if $\lambda_{(G,f)}$ and $\lambda_{(G',f')}$ are stable then

$$d\big(D(f), D(f')\big) \leq \delta\big((G, f), (G', f')\big)$$

Stability

Related with stability, we have the problem of universality: Is the inequality of Thm. 3.3 the best one that we can obtain from persistence diagrams?

This issue can only be addressed by *ad hoc* constructions.

Coherent samplings

A first way of building persistence functions is the following.

Definition 3.4 (Coherent sampling)

A *coherent sampling* $\mathcal{V}$ is the assignment to each graph G, where $G = (V, E)$ of a set $\mathcal{V}(G)$ of subsets of $V \cup E$, such that the following conditions 1 and 2 hold; it will be said to be a *stable* coherent sampling if also condition 3 holds:

1. each $\mathcal{V}(G)$ is finite (possibly empty);
2. if G is a subgraph of H, then each element of $\mathcal{V}(G)$ is contained in exactly one element of $\mathcal{V}(H)$;
3. if $\psi : G \to G'$ is an isomorphism, then $\mathcal{V}(G') = \psi\big(\mathcal{V}(G)\big)$.

For each inclusion $\iota : G \to H$ let $\Lambda(\iota)$ be the number of elements of $\mathcal{V}(H)$ containing at least one element of $\mathcal{V}(G)$.

Coherent samplings

Proposition 3.5

Let a coherent sampling $\mathcal{V}$ be given; for all graphs $G = (V, E)$, for all filtering functions $f : E \to \mathbb{R}$, let $\lambda_{(G,f)} : \Delta^+ \to \mathbb{Z}$ be defined by $\lambda_{(G,f)}(u, v) = \Lambda(\iota_{(u,v)})$ where $\iota_{(u,v)} : G_u \to G_v$ is the inclusion homomorphism.
Then the functions $\lambda_{(G,f)}$ are persistence functions. If the coherent sampling is stable, so are the persistence functions.

Example 1: Blocks

We recall that in a (loopless) graph G a *cut vertex* (or *separating vertex*) is a vertex $v \in V(G)$ whose deletion (along with incident edges) makes the number of connected components of G increase. A *block* is a connected graph which does not contain any cut vertex. A block of a graph G is a maximal subgraph H such that H is a block.

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Proposition 3.6

The assignment $\mathcal{B}$, which maps each graph G to the set of its blocks, is a stable coherent sampling.

Example 1: Blocks

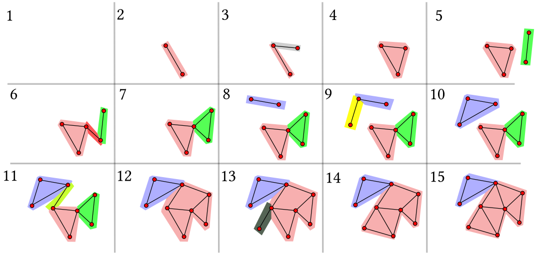
Definition 3.7

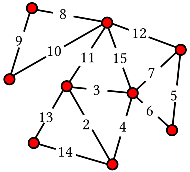
Given a weighted graph (G, f), we call *persistent block number* the function $bl_{(G,f)} : \Delta^+ \to \mathbb{Z}$ which maps the pair (u, v) to the number of blocks of G_v containing at least one block of G_u.

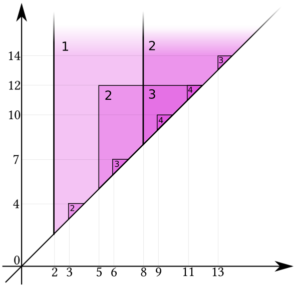
Corollary 3.8

$bl_{(G,f)}$ is a stable persistence function.

Example 1: Blocks







Example 1: Blocks

We can then associate to $bl_{(G,f)}$ a *persistent block diagram* $D_{bl}(f)$ with all classical features.

Theorem 3.9 (Universality)

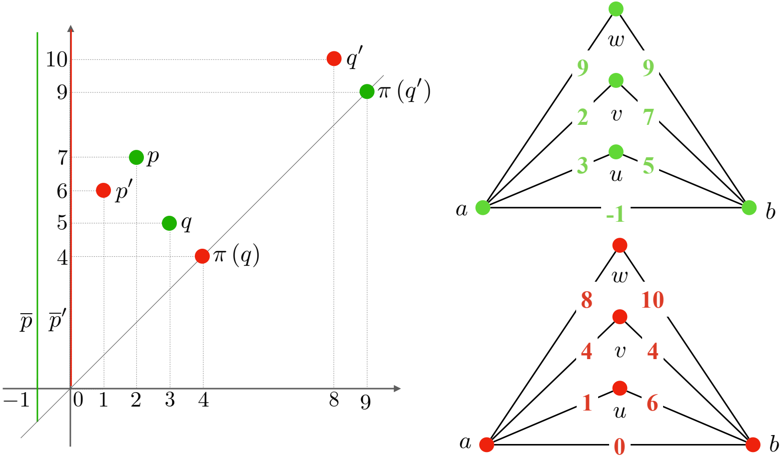
If $\tilde{d}$ is a distance for persistent block diagrams such that

$$\tilde{d}\big(D_{bl}(f), D_{bl}(f')\big) \leq \delta\big((G, f), (G', f')\big)$$

for any persistent block diagrams $D_{bl}(f)$, $D_{bl}(f')$ of weighted graphs (G, f), (G', f'), with G, G' isomorphic, then

$$\tilde{d}\big(D_{bl}(f), D_{bl}(f')\big) \leq d\big(D_{bl}(f), D_{bl}(f')\big)$$

Example 1: Blocks



Example of the construction needed for universality of *bl*.

Example 2: Edge-blocks

We recall that in a graph G a *cut edge* (or *bridge*) is an edge $e \in E(G)$ whose deletion makes the number of connected components of G increase. We define an *edge-block* as a connected graph which contains at least one edge, but does not contain any cut edge. An edge-block of a graph G is a maximal subgraph H such that H is an edge-block.

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Proposition 3.10

The assignment $\mathcal{E}$, which maps each graph G to the set of its edge-blocks, is a stable coherent sampling. □

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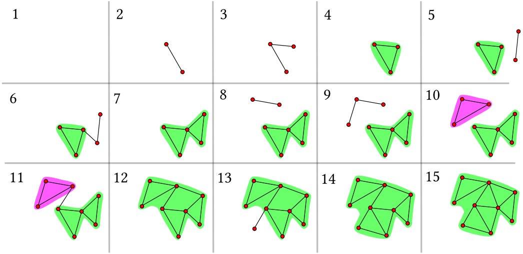
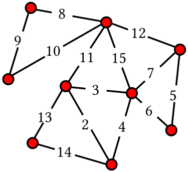
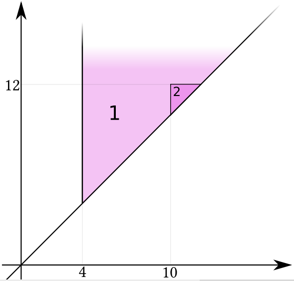
The definition of a *persistent edge-block number* function $ebl_{(G,f)}$, its stability and universality also hold.

Example 2: Edge-blocks







Steady and ranging

Given a graph $G = (V, E)$, let $F : 2^{V \cup E} \to \{\mathit{true}, \mathit{false}\}$ be any feature. We call *F-set* any set $A \subseteq V \cup E$ such that $F(A) = \mathit{true}$.

Let now the weighted graph (G, f) be given. Given any real number w, we shall say that $A \subseteq V \cup E$ is an F-set *at level w* if it is an F-set of the subgraph G_w.

Steady and ranging

Definition 3.11 (Steady and ranging)

We call $A \subseteq V \cup E$ a *steady* F-set (or simply an s-F-set) at (u, v) $((u, v) \in \Delta^+)$ if it is an F-set at all levels w with $u \leq w \leq v$.
We call A a *ranging* F-set (or simply an r-F-set) at (u, v) if there exist levels $w \leq u$ and $w' \geq v$ at which it is an F-set.

Let $SF_{(G,f)}(u, v)$ be the set of s-F-sets at (u, v) and let $RF_{(G,f)}(u, v)$ be the set of r-F-sets at (u, v).

Steady and ranging

Proposition 3.12

The function which assigns to $(u, v) \in \Delta^+$ the number $|SF_{(X,f)}(u, v)|$ is a persistence function.

Proposition 3.13

The function which assigns to $(u, v) \in \Delta^+$ the number $|RF_{(X,f)}(u, v)|$ is a persistence function.

Example: Eulerian sets

Given any graph $G = (V, E)$, we define $Eu : 2^{V \cup E} \rightarrow \{true, false\}$ to yield *true* on a set A if and only if A is a set of vertices whose induced subgraph of G is nonempty, connected, Eulerian and maximal with respect to these properties; in that case A is said to be a *Eu*-set of G.

Let now (G, f) be a weighted graph. We apply Def. 3.11 to feature Eu for a weighted graph (G, f).

Example: Eulerian sets

Definition 3.14

Given any real number w, the set of vertices A is a *Eu*-set *at level* w if it is a *Eu*-set of the subgraph G_w.

It is a *steady* *Eu*-set (an s-*Eu*-set) at (u, v) ($(u, v) \in \Delta^+$) if it is a *Eu*-set at all levels w with $u \leq w \leq v$.

It is a *ranging* *Eu*-set (an r-*Eu*-set) at (u, v) if there exist levels $w \leq u$ and $w' \geq v$ at which it is a *Eu*-set.

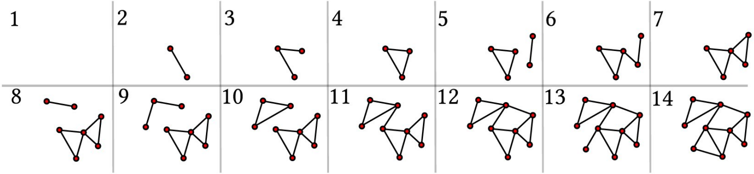
$SEu_{(G,f)}(u, v)$ and $REu_{(G,f)}(u, v)$ are respectively the sets of s-*Eu*-sets and of r-*Eu*-sets at (u, v).

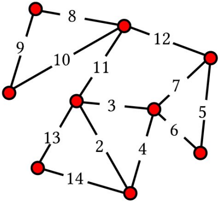
Example: Eulerian sets

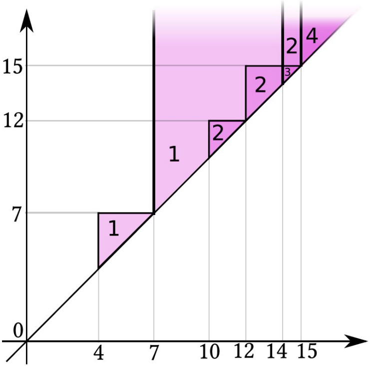
Proposition 3.15

The function σeu *which assigns to* $(u, v) \in \Delta^+$ *the number* $|SEu_{(G,f)}(u, v)|$ *and the function* ϱeu *which assigns to* $(u, v) \in \Delta^+$ *the number* $|REu_{(G,f)}(u, v)|$ *are persistence functions.*

Example: Eulerian sets







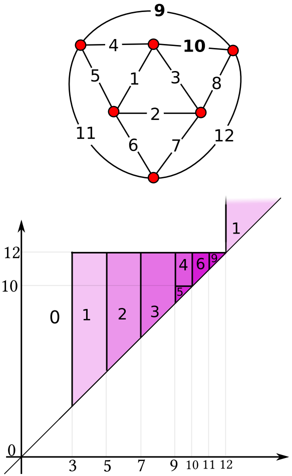
Example: Eulerian sets

Both functions can be proved to be unstable.

The example of next figure shows that the function σeu is not stable: In fact, the maximum absolute value of the weight difference on the same edges is 1, and $\sigma_{(G,f)}(4.5-1, 10+1) = 1 > 0 = \sigma_{(G,g)}(4.5, 10)$, against Condition 3 of Def. 3.1.

The instability of ϱeu is proved analogously.

Example: Eulerian sets



Instability of σeu: Filtering function f left, g right.

Categorical persistence functions

Coherent samplings are a direct generalization of 0-PBNs. The nature of general k-PBNs has suggested a still wider generalization[2]: rank-based persistence, of which k-PBNs and persistence functions are particular classes of examples.

[2] Bergomi, M.G., Vertechi, P., *Rank-based persistence* (2019), preprint available at https://arxiv.org/abs/1905.09151

Categorical persistence functions

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Definition 4.1 (Categorical persistence function)

Let **D** be a category. We say that a lower-bounded function $p : \mathrm{Morph}(\mathbf{D}) \to \mathbb{Z}$ is a *categorical persistence function* if, for all $u_1 \to u_2 \to v_1 \to v_2$, the following inequalities hold:

1. $p(u1 \to v1) \leq p(u2 \to v1)$ and $p(u2 \to v2) \leq p(u2 \to v1)$.
2. $p(u2 \to v1) - p(u1 \to v1) \geq p(u2 \to v2) - p(u1 \to v2)$.

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$(\mathbb{R}, \leq)$-indexed diagrams

Also the notion of filtered object (which already generalizes the one of size pair (X, f) we have dealt with so far) can be widened.

Definition 4.2 ($(\mathbb{R}, \leq)$-indexed diagram)

Given a category **C**, a $(\mathbb{R}, \leq)$-*indexed diagram in* **C** is a functor from $(\mathbb{R}, \leq)$ to **C**.

Since $\overline{\Delta}^+$ is in 1-1 correspondence with $\mathrm{Morph}(\mathbb{R}, \leq)$, given a categorical persistence function on **C** and an $(\mathbb{R}, \leq)$-indexed diagram in **C**, we get an integer valued function on $\overline{\Delta}^+$ with the same features as a persistence function (and as PBNs), and the possibility to condense information in a persistence diagram.

Rank functions

In classical persistence, dimension (of homology modules and submodules) plays a central role. This can be generalized too.

Definition 4.3 (Rank function)

Let $\mathbf{R}$ be an Abelian category. Given a lower-bounded function $r : \mathrm{Obj}(\mathbf{R}) \to \mathbb{Z}$, we say that r is a *rank function* if:

1. For any monomorphism $A \hookrightarrow B$, $r(A) \leq r(B)$
2. For any epimorphism $B \twoheadrightarrow D$, $r(B) \geq r(D)$
3. For all short exact sequence $A \hookrightarrow B \twoheadrightarrow D$, $r(A) + r(D) = r(B) + r(0)$

A ranked category $(\mathbf{R}, r)$ is an Abelian category $\mathbf{R}$ equipped with a rank function r.

Rank functions

The definition can actually be given in the wider context of “regular” categories.

Proposition 4.4

Given a ranked category $(\mathbf{R}, r)$ *and a functor* $F : \mathbf{C} \to \mathbf{R}$*, the function* $r \circ im \circ F$ *is a categorical persistence function on* $\mathbf{C}$*.*

Rank functions

Figure 3: From the classical to the categorical framework.

Classical framework	**Categorical framework**
Topological spaces	Source category **C**
Vector spaces	Regular target category **R**
Dimension	Rank function on **R**
Homology functor	Arbitrary functor from **C** to **R**
Filtration of topological spaces	$(\mathbb{R}, \leq)$-indexed diagram in **C**

END OF THE THIRD PART

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All our software concerning graph persistence can be found at
`https://gitlab.com/mattia.bergomi/perscomb`

Non-topological persistence - Applications

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[1] Dip. di Matematica and ARCES, Univ. di Bologna, Italia (Vision Mathematics group)

Work performed in collaboration with

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Outline

1. **Applications of steady and ranging**
2. **Cornerpoint selection**
3. **Conclusions**

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2. Cornerpoint selection
3. Conclusions

Steady and ranging hubs

We now apply the persistence functions coming from ranging and steady sets, together with a smart idea of V. Kurlin, to the study of "hubs" in networks. As always, (G, f) is given, with $G = (V, E)$.

The property we are going to use gives *false* for all subsets of $V \cup E$ apart from the singletons formed by vertices whose degree is greater than or equal to the degree of all their neighbors:

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Definition 1.1

A *temporary hub (t-hub) at level u* is a vertex of G_u whose degree is greater than or equal to the degree of its neighbors.

Steady and ranging hubs

Definition 1.2

A *steady hub (s-hub) at* (u, v) ($(u, v) \in \Delta^+$) is a vertex which is a t-hub at all levels w with $u \leq w \leq v$.

Definition 1.3

A *ranging hub (r-hub) at* (u, v) ($(u, v) \in \Delta^+$) is a vertex such that there exist levels $w \leq u$ and $w' \geq v$ at which it is a t-hub.

Steady and ranging hubs

We define $\sigma_{(G,f)} : \Delta^+ \to \mathbb{Z}$ as follows: For every $(u, v) \in \Delta^+$, $\sigma_{(G,f)}(u, v)$ is the number of s-hubs at (u, v).

Proposition 1.4

σ *is a persistence function.*

We define $\varrho_{(G,f)} : \Delta^+ \to \mathbb{Z}$ as follows: For every $(u, v) \in \Delta^+$, $\varrho_{(G,f)}(u, v)$ is the number of r-hubs at (u, v).

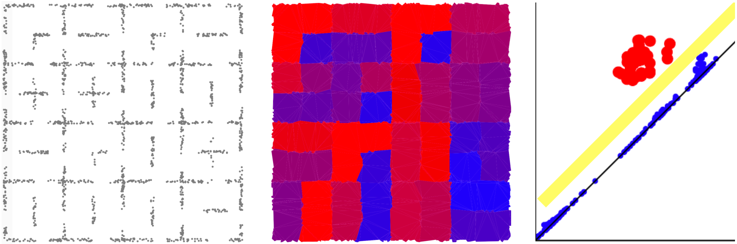
Proposition 1.5

ϱ *is a persistence function.*

Both functions can be proved to be **unstable**.

Hub selection

Finally, we use Kurlin's *widest diagonal gap* for selecting the top s- and r-hubs.



V. Kurlin, *A fast persistence-based segmentation of noisy 2D clouds with provable guarantees* Pattern Recognition Letters, 83 (2016), 3–12

Example 1: airports

From now on I report WORK IN PROGRESS, not necessarily as satisfatory as we could wish . . ..

A first application of the search for relevant hubs has been done on a set of major North-American airports. The edges connect airports between which there are regular flights.

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As filtering functions we use:

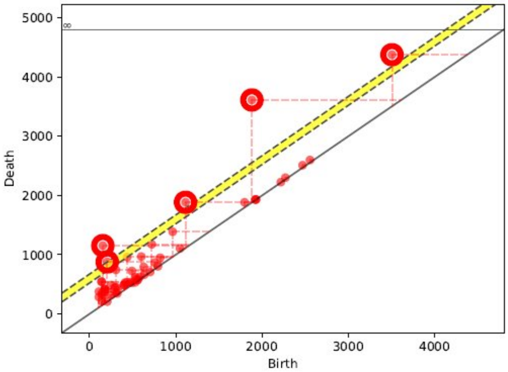
- distance
- weekly flight frequency
- their product

and their opposites (+their maximum)

Example 1: airports

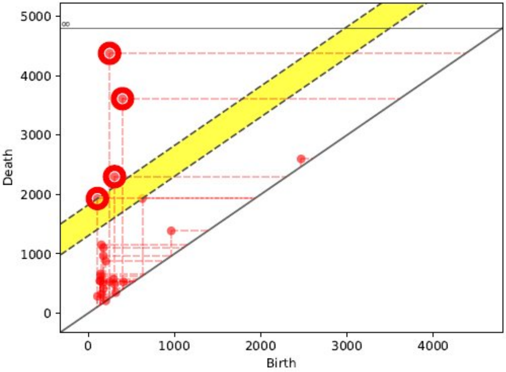
Albuquerque	Atlanta	Baltimore	Boston	Buffalo
Cheyenne	Chicago	Cincinnati	Cleveland	Dallas
Denver	Detroit	El Paso	Houston	Indianapolis
Jacksonville	Kansas City	Las Vegas	Los Angeles	Memphis
Miami	Milwaukee	Mobile	Montreal	New Orleans
New York	Oakland/Emeryville	Philadelphia	Phoenix	Pittsburgh
Portland	Sacramento	Salt Lake City	San Antonio	San Diego
San Francisco	Seattle	St. Louis	St. Paul-Minneapolis	Tampa
Toronto	Tucson	Vancouver	Washington	

Example 1: airports (distance)



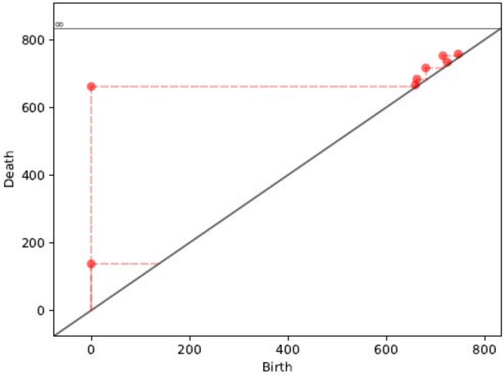
Steady hubs above 2nd widest gap							
5 cornerpoints, 4 vertices							
Atlanta	2	Dallas	1	Detroit	1	Seattle	1

Example 1: airports (distance)



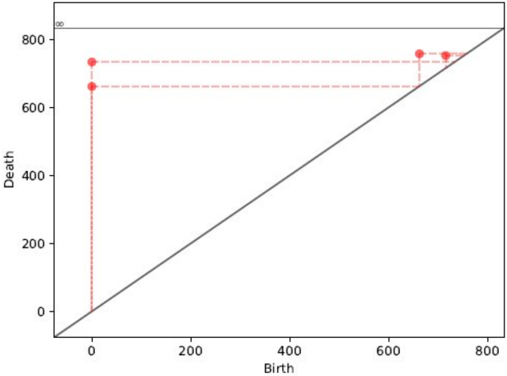
Ranging hubs above 3rd widest gap			
4 cornerpoints, 4 vertices			
Atlanta	Chicago	Dallas	Houston

Example 1: airports (max - frequency)



Steady hubs							
8 cornerpoints, 4 hubs							
Atlanta	2	Chicago	4	Dallas	1	New York	1

Example 1: airports (max - frequency)



Ranging hubs			
4 cornerpoints, 4 hubs			
Atlanta	Chicago	Dallas	New York

Example 2: European languages

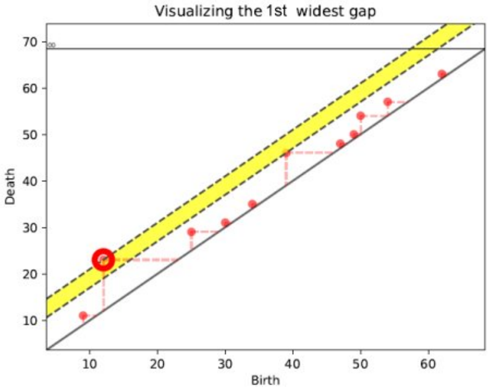
A second application is on languages of the European Union plus Turkish:

Castilian	Catalan	Croatian	Czech	Danish
Dutch	English	Finnish	French	Galitian
German	Greek	Hungarian	Italian	Polish
Portuguese	Romanian	Swedish	Turkish	

The graph is complete.

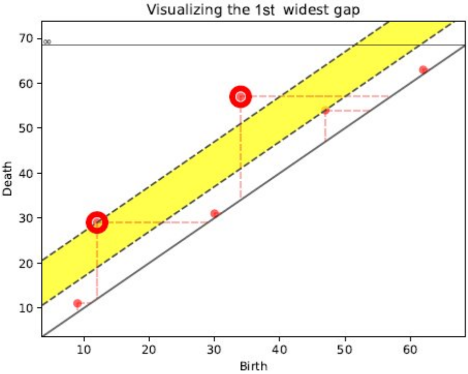
Filtering function is the opposite of the percentage of common properties (+ its max).

Example 2: European languages



Steady hubs					
11 cornerpoints, 6 vertices					
Castilian	1	Catalan	2	Dutch	1
English	2	Portuguese	4	Swedish	1

Example 2: European languages

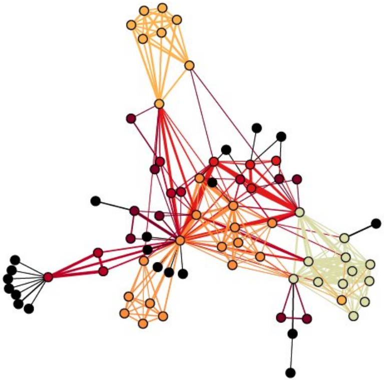


Ranging hubs		
6 cornerpoints, 6 vertices		
Castilian English	Catalan Portuguese	Dutch Swedish

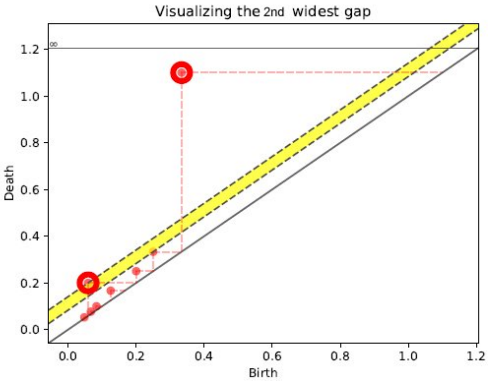
Example 3: Les Misérables

We now study the network formed by the 77 main characters of *Les Misérables*, with 254 edges, each representing the simultaneous presence of the incident vertices in at least one scene.

Each edge is labeled with the number of such scenes. As filtering function we take the inverse of this number.

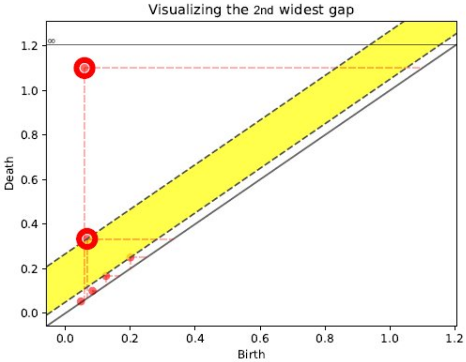


Example 3: Les Misérables



Steady hubs					
8 cornerpoints, 6 vertices					
Cosette	1	Courfeyrac	1	Enjolras	2
Marius	1	Myriel	1	Valjean	2

Example 3: Les Misérables



Ranging hubs		
6 cornerpoints, 6 vertices		
Cosette	Courfeyrac	Enjolras
Marius	Myriel	Valjean

Example 4: Set persistence: a non-graph example

Let a grey-tone image be given, as a function $f : D \to \mathbb{R}$, where $D = \{1, \ldots, M\} \times \{1, \ldots, N\} \subset \mathbb{Z}^2$.

Fix a positive integer k. We define the neighbor set of a pixel $x = (i, j)$ as

$$N_k(x) = \{x' = (i', j') \in D \mid i' = i + m, \;\; j' = j + n, \;\; -k \leq m, n \leq k\}$$

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Definition 1.6

Fix positive integers m, n . Set $D' = \{x \in D \,|\, |N_k(x)| > m + n\}$. We say that a pixel $x \in D'$ is active at level $u \in \mathbb{R}$ (briefly an A-pixel at level u) if the following conditions are satisfied

1. $|N_k(x) \cap f^{-1}([-\infty, u])| \geq m$
2. $|N_k(x) - f^{-1}([-\infty, u])| \leq n$

Example 4: Set persistence: a non-graph example

Note that $|N_k(x) \cap f^{-1}([-\infty, u])|$ is monotonically non-decreasing with u, so (the singleton formed by) a pixel is a ranging-A-pixel at (u, v) if and only if it is a steady-A-pixel at (u, v).

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Let $\alpha(u, v)$ be the number of steady-A-pixels at (u, v).

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Proposition 1.7

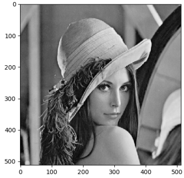
α *is a persistence function.*

Example 4: Set persistence: a non-graph example

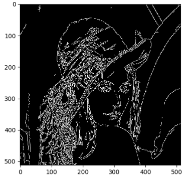
Now, for a given image we can build the persistence diagram relative to α and pick up cornerpoints $(\bar{u}, \bar{v})$ whose persistence (also called lifetime) $\bar{v} - \bar{u}$ exceeds a given threshold t. The sets of steady-A-pixels at those pairs $(\bar{u}, \bar{v})$ may build reasonable contours for the image.

This is what we have done in the next examples, compared with the classical Canny algorithm, for images with more and more noise.

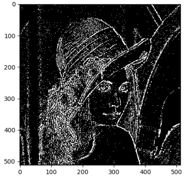
Persistent Edge Detection



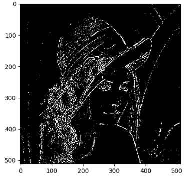
(a) Original



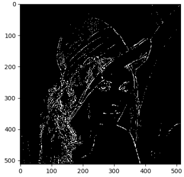
(b) Canny



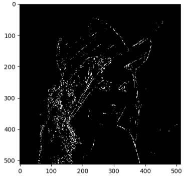
(c) Ours: $t = 5$



(d) Ours: $t = 10$

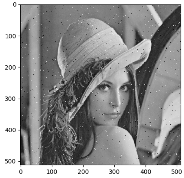


(e) Ours: $t = 15$

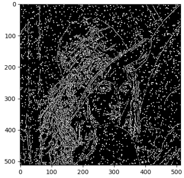


(f) Ours: $t = 20$

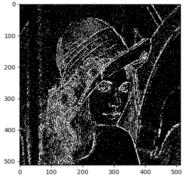
Persistent Edge Detection



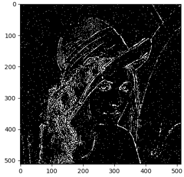
(a) Original



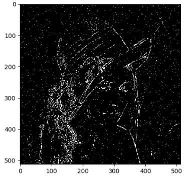
(b) Canny



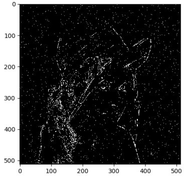
(c) Ours: $t = 5$



(d) Ours: $t = 10$

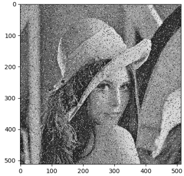


(e) Ours: $t = 15$

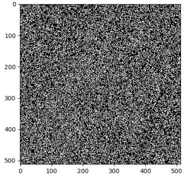


(f) Ours: $t = 20$

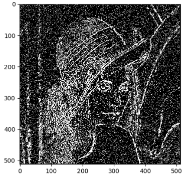
Persistent Edge Detection



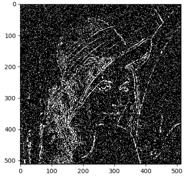
(a) Original



(b) Canny



(c) Ours: $t = 5$



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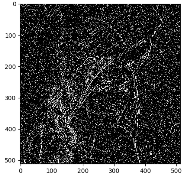
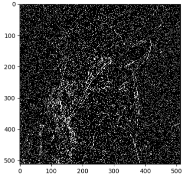


(e) Ours: $t = 15$



(f) Ours: $t = 20$

1. Applications of steady and ranging
2. **Cornerpoint selection**
3. Conclusions

Ziqqurat

We are developing a new (somewhat self-referential) selection method for the cornerpoints of a given persistence diagram.

Ziqqurat

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The idea is to rank clusters of cornerpoints and to select one cornerpoint out of each cluster. To do so, we build a polyhedron ("ziqqurat") with steps, whose corners correspond to cornerpoints, so that all corners of the ziqqurat belong to a same plane. We then filter the ziqqurat moving that plane parallel to itself.

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We so form a rather poor persistence diagram: all cornerpoints are on the same vertical line. On that line the rank is given by ordinate.

Ziqqurat

Let D be a persistence diagram with a finite set of cornerpoints; assume all multiplicities equal to 1. Also assume that all points on the diagonal Δ belong to D. Also add a cornerpoint at $(-\infty, +\infty)$.

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For every $(\overline{u}, \overline{v}) \in \mathbb{R}^2$ we define the legacy $\eta\big((\overline{u}, \overline{v})\big) =$

$$\max\{\tilde{u} - \tilde{v} \mid (\tilde{u}, \tilde{v}) \in D, \ \ \tilde{u} \leq \overline{u}, \ \ \tilde{v} \geq \overline{v}\}$$

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Now define the ziqqurat Z_D as

$$Z_D = \{(u, v, w) \in \mathbb{R}^3 \mid \eta\big((u, v)\big) > -\infty, \ \ w \leq \eta\big((u, v)\big)\}$$

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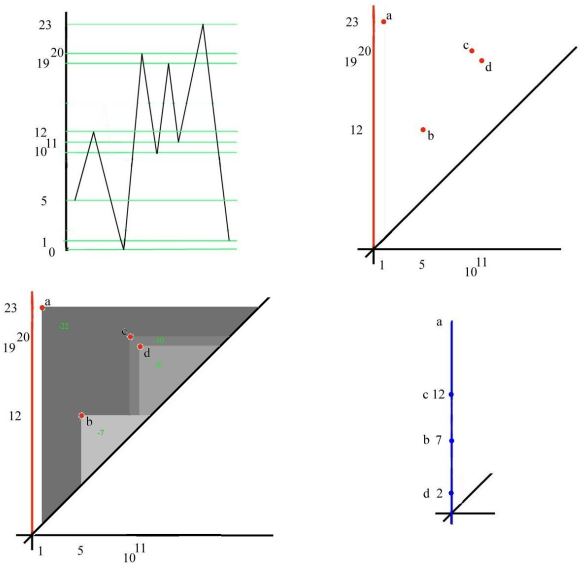
Now we can consider the persistence diagram $\mathcal{D}$ of the pair (Z_D, f) with filtering function $f = u - v - w$.

Ziqqurat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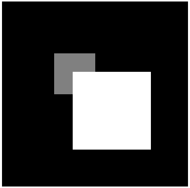
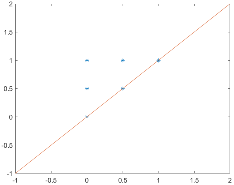
b 7
d 2

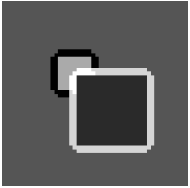
Ziqqurat



(a) Original



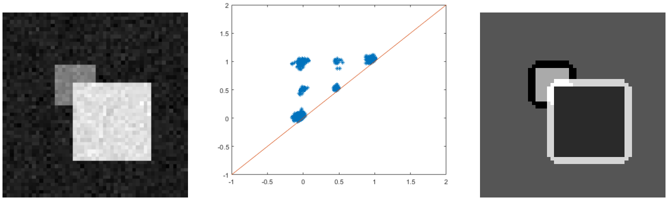
(b) Persistence diagram



(c) Segmentation

The result with kernel size $k_s = 1$, $m = 2$, $n = 7$

Ziqqurat



(a) Original **(b)** Persistence diagram **(c)** Segmentation

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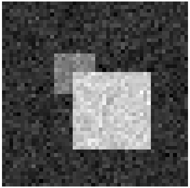
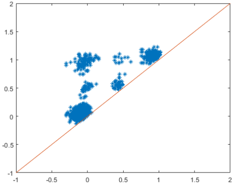
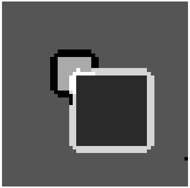


(a) Original



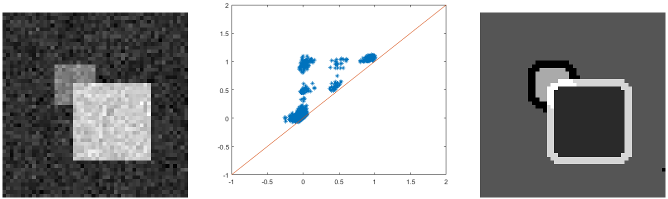
(b) Persistence diagram



(c) Segmentation

The result with kernel size $k_s = 1$, $m = 2$, $n = 7$

Ziqqurat



(a) Original **(b)** Persistence diagram **(c)** Segmentation

Figure 7: The result with kernel size $k_s = 2$, $m = 8$, $n = 18$

1. Applications of steady and ranging
2. Cornerpoint selection
3. **Conclusions**

Signatures of data by persistence diagrams are possible also without the need of topological constructions.

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Although most of our examples are in graph theory, the categorical setting promises an extension to a much wider context.

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Here we have presented tentative applications of two general construction methods for persistence functions:

- Coeherent samplings
- Steady and ranging sets.

THANKS FOR YOUR ATTENTION

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lorenzo.zuffi@studio.unibo.it

All our software concerning graph persistence can be found at
`https://gitlab.com/mattia.bergomi/perscomb`