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EXPLICITLY SOLVABLE OPTIMAL DISCRETE MODELS WITH CONTROLLED DISBALANCE OF THE TOTAL MECHANICAL ENERGY FOR DYNAMICAL PROBLEMS OF LINEAR ELASTICITY

© **A. N. Konovalov and Yu. P. Popov** UDC 519.63:539.3

Abstract: Considering the dynamical problems of linear elasticity, we construct and justify explicitly solvable discrete (mesh) models with controlled disbalance of the total mechanical energy and maximally possible parallelism degree.

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The mathematical modeling of a physical process on a computer is connected with a technological chain [1, Chapter 1] one of whose steps can be represented as follows: $I \to I_h \to II \to III$. Here I is a contiguous mathematical model of the process, I_h is a discrete model, II is an algorithm for the realization of the discrete model, and III is a program for a particular computer. In what follows, we confine exposition to the optimization of the step $I \to I_h \to II$ for dynamical problems of linear elasticity.

Assign the phase volume $G = [0 \le t \le t_*] \times V$ to the process under consideration. Here t is time, and, in the Lagrange description, as $V = V(M)$ we can take a fixed space volume $V(M)$ with boundary $\gamma = \gamma_1 \cup \gamma_2 = \partial V$. Thus, the parameters of the process are defined as tensor functions of the vector argument (M, t), $M \in V$. For a dynamical problem, the displacement vectors $u(M, t)$ and the velocity $v(M, t)$ are treated as elements of a Hilbert space $H^1(t)$ and the rank-two strain and stress tensors $\varepsilon(M, t)$ and $\sigma(M, t)$ as elements of a Hilbert space $H^2(t)$. The inner products on $H^1(t)$ and $H^2(t)$ are defined as the convolutions of the corresponding elements [2, Chapter II]. The connection between the parameters of the dynamical problem is given by the state equation, the defining relations, and the impulse conservation law in the field of mass forces. Put (here T stands for matrix transposition)

$$\sigma_1 = (\sigma_{11}, \sigma_{22}, \sigma_{33})^T, \quad \sigma_2 = (\sigma_{12}, \sigma_{13}, \sigma_{23})^T, \quad \sigma = (\sigma_1, \sigma_2)^T,$$
$$\varepsilon_1 = (\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33})^T, \quad \varepsilon_2 = (2\varepsilon_{12}, 2\varepsilon_{13}, 2\varepsilon_{23})^T, \quad \varepsilon = (\varepsilon_1, \varepsilon_2)^T.$$

Then the state equation

$$\sigma = K\varepsilon \leftrightarrow \sigma_{ij} = \mu 2\varepsilon_{ij} + \lambda\delta_{ij}(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) \tag{1}$$

is defined by the block matrix $K : H^2 \to H^2$, where

$$K = \begin{pmatrix} K_1 & 0 \\ 0 & K_2 \end{pmatrix}, \; K_1 = \begin{pmatrix} (\lambda+2\mu) & \lambda & \lambda \\ \lambda & (\lambda+2\mu) & \lambda \\ \lambda & \lambda & (\lambda+2\mu) \end{pmatrix}, \; K_2 = \begin{pmatrix} \mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu \end{pmatrix}.$$

Here $\lambda(M) > 0$ and $\mu(M) > 0$ are the Lamé coefficients, $K = K^T > 0$, and (1) can be written down as follows: $\sigma_1 = K_1\varepsilon_1$ and $\sigma_2 = K_2\varepsilon_2$.

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If $u = (u_1, u_2, u_3)^T$, $v = (v_1, v_2, v_3)^T$, and $\partial u/\partial t = v$, then

$$\varepsilon = Ru = \begin{pmatrix} \partial_1 & 0 & 0 \\ 0 & \partial_2 & 0 \\ 0 & 0 & \partial_3 \\ \partial_2 & \partial_1 & 0 \\ \partial_3 & 0 & \partial_1 \\ 0 & \partial_3 & \partial_2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \leftrightarrow 2\varepsilon_{ij} = \partial_j u_i + \partial_i u_j, \quad \partial_i = \frac{\partial}{\partial x_i}, \tag{2}$$

and the operator $R : H^1 \to H^2$ is defined. We will assume that if $u \in H^1$ and $\sigma \in H^2$ then

$$u_{/\gamma_1} = 0, \quad \sigma_{ij} n_{j\,/\gamma_2} = 0, \tag{3}$$

where n_j are the components of the unit outer normal to γ_2. For the operator $R^* : H^2 \to H^1$ adjoint to R, by definition,

$$(u, R^*\sigma)_{H^1} = (Ru, \sigma)_{H^2}, \quad u \in H^1, \ \sigma \in H^2.$$

Therefore,

$$R^*\sigma = -R^T\sigma = -\begin{pmatrix} \partial_1 & 0 & 0 & \partial_2 & \partial_3 & 0 \\ 0 & \partial_2 & 0 & \partial_1 & 0 & \partial_3 \\ 0 & 0 & \partial_3 & 0 & \partial_1 & \partial_2 \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} = -\operatorname{div}\sigma, \tag{4}$$

and, reckoning with (3), we can write down the impulse conservation law as

$$\rho\frac{\partial v}{\partial t} + R^*\sigma = \rho f. \tag{5}$$

Thus, we come to the closed "continuous" model for a linear dynamical problem of elasticity:

$$\begin{gathered} \rho\frac{\partial v}{\partial t} + R^*\sigma = \rho f(M), \quad \varepsilon = Ru, \quad \sigma = K\varepsilon, \quad \frac{\partial u}{\partial t} = v, \\ u(M,t) = \varphi_1(M), \quad v(M,0) = \varphi_2(M). \end{gathered} \tag{6}$$

The boundary conditions (3) must be taken into account when the domains of R and R^* are defined.

In (6), there is no prescribed order for the determination of the desired parameters and, alongside (6), also other closed models of the problem under study can be considered, for example, "in displacements." Then

$$\rho\frac{\partial v}{\partial t} + R^*\sigma = \rho f \to \rho\frac{\partial^2 u}{\partial t^2} + R^*K\varepsilon = \rho f \to \rho\frac{\partial^2 u}{\partial t^2} + R^*KRu = \rho f. \tag{7}$$

However, the last passage in (7) is connected with the tacit assumption of the solvability of the operator equation

$$Ru(M,t) = \varepsilon(M,t) \tag{8}$$

for every $t > 0$. We will use the condition for solvability of (8) in the following form [3, Chapter IV]:

$$(\varepsilon, \psi)_{H^2} = 0, \quad R^*\psi = 0. \tag{9}$$

We can show that the Saint-Venant compatibility (continuity) conditions, the equality of the incompatibility tensor to zero, and (9) are equivalent.

Lemma 1. *If $\partial u/\partial t = v \in H^1$ and*

$$\frac{\partial \varepsilon}{\partial t} - Rv = 0, \quad \varepsilon(M,0) = R\varphi_1(M) \tag{10}$$

for $\varepsilon(M,t) \in H^2$, then the operator equation (8) *is solvable for every $t > 0$ and $u \in H^1$ is determined from* (8) *up to the kernel of R: $u = R^{-1}\varepsilon + u_*$, $Ru_* = 0$. The element u_* (the rigid displacement vector) is fixed by the boundary condition γ_1.*

The use of the defining relation (2) in the form (10) makes it possible to pass from (6) to the closed model

$$D\frac{\partial w}{\partial t} + Aw = f_*, \quad w = (v,\sigma)^T, \quad f_* = (\rho f, 0)^T, \quad \frac{\partial u}{\partial t} = v, \quad \varepsilon = K^{-1}\sigma,$$
$$D = \begin{pmatrix} \rho E_1 & 0 \\ 0 & K^{-1}E_2 \end{pmatrix}, \ E_1 v = v \in H^1, \ E_2\sigma = \sigma \in H^2, \ A = \begin{pmatrix} 0 & R^* \\ -R & 0 \end{pmatrix}, \tag{11}$$

with a skew-symmetric operator A, the initial data from (6), (10), and the boundary conditions (3). In (11), we must firstly find the covector w and only then the parameters $u \in H^1$, $\varepsilon \in H^2$.

Theorem 1. *The basic model* (11) *admits the additional conservation law*

$$\frac{\partial J(t)}{\partial t} = (\rho f, v)_{H^1}, \quad J(t) = J_1(t) + J_2(t) = 0.5(\rho v, v)_{H^1} + 0.5(\varepsilon,\sigma)_{H^2}. \tag{12}$$

Rewrite (12) as

$$J(t+\Delta t) = J(t) + \int\limits_t^{t+\Delta t} \left(\rho f, \frac{\partial u}{\partial t}\right)_{H^1} dt = J(t) + Q(t+\Delta t). \tag{13}$$

Note that, in (12), $J_1(t)$ corresponds to the kinetic energy of the elastic medium, $J_2(t)$ corresponds to the potential energy of elastic deformations related to the volume V, and Q in (13) is the work of the mass forces on the increments Δu at the time Δt. Therefore, (12) is usually referred to as the conservation law of the total mechanical energy of the elastic medium in the volume V.

Turn to the passage $I \to I_h$. The notations below are standard in the theory of difference schemes. Thus,

$$G \to G_h = [0,\tau,\dots,n\tau,\dots,k\tau = t_*] \times V_h, \ M_h \in V_h, \ w_h^n = w(M_h, n\tau) = \left(v_h^n, \sigma_h^n\right)^T,$$
$$\gamma \to \gamma_h = \gamma_{1h} \cup \gamma_{2h}, \ u_{/\gamma_1} = 0 \to u^n_{h/\gamma_{1h}} = 0, \ \sigma_{ij} n_{j/\gamma_2} = 0 \to \left(\sigma_h^n\right)_{ij} n_{j/\gamma_{2h}} = 0,$$
$$\tau(\cdot)_{\bar t}^{n+1} = (\cdot)_h^{n+1} - (\cdot)_h^n, \ \partial_j^+(\cdot)_h = (\cdot)_{x_j}, \ \partial_j^-(\cdot)_h = (\cdot)_{\bar x_j}, \ (\cdot)_\alpha = \alpha(\cdot)_h^{n+1} + (1-\alpha)(\cdot)_h^n.$$

At the passage $I \to I_h$, the main requirement is that of the adjoint-coordinated approximation [4]:

$$(Ru,\sigma)_{H^2} = (u, R^*\sigma)_{H^1} \to (R_h u_h, \sigma_h)_{H_h^2} = (u_h, R_h^*\sigma_h)_{H_h^1}. \tag{14}$$

This requirement is rather constructive, and if, for example, in $R = R(\partial_1,\partial_2,\partial_3)$, for the operator ∂_j, we choose an approximation ∂_j^+ then, for the same operator ∂_j in $R^* = R^*(\partial_1,\partial_2,\partial_3)$, we must choose the approximation ∂_j^-.

The main computational problems at the passage $I \to I_h$ are connected with the quality of the difference scheme for the problem

$$D\frac{\partial w}{\partial t} + Aw = f_*, \quad w(M,0) = (\varphi_2(M), KR\varphi_1(M))^T. \tag{15}$$

To the continuous problem (15), assign the discrete adjoint-coordinated two-parameter model

$$\rho_h v_{\bar t}^{n+1} + R_h^*\sigma_\alpha = \rho_h f_h, \quad K_h^{-1}\sigma_{\bar t}^{n+1} - R_h v_\beta = 0, \quad v_h^0 = \varphi_{2h}, \quad \varepsilon_h^0 = R_h\varphi_{1h},$$

for which we will use the operator representation: $y = (v_h, \sigma_h)^T$,

$$B_h y_{\bar t}^{n+1} \equiv (D_h + \tau A_{\alpha,\beta}) y_{\bar t}^{n+1} = (f_*)_h - A_h y^n \equiv r^n, \tag{16}$$

where the element y^0 is given. Here

$$D_h = \begin{pmatrix} \rho_h E_1 & 0 \\ 0 & K_h^{-1} E_2 \end{pmatrix}, \quad A_{\alpha,\beta} = A_\alpha + A_\beta = \begin{pmatrix} 0 & \alpha R_h^* \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -\beta R_h & 0 \end{pmatrix},$$

$$A_h = A_{1,1}.$$

In (16), the general principles of constructing economical algorithms for the inversion of the operator B_h in the passage $y^n \to y^{n+1}$ are rather well known. Here we will utilize the method of approximate factorization and, instead of (16), in the passage $y^n \to y^{n+1}$ we will use the factorized discrete model [5, Chapter XIV]

$$(B_*)_h y_{\bar{t}}^{n+1} = r^n, \quad (B_*)_h = (D_h + \tau A_\alpha) D_h^{-1} (D_h + \tau A_\beta), \tag{17}$$

where the element y^0 is given. The formal ground for such a change is as follows: $(B_*)_h = B_h + O(\tau^2)$. The realization of (17) is carried out in the standard manner. So,

$$\begin{pmatrix} \rho_h E_1 & \alpha\tau R^* \\ 0 & K_h^{-1} E_2 \end{pmatrix} \begin{pmatrix} v_* \\ \sigma_* \end{pmatrix} = \begin{pmatrix} r_1^n \\ r_2^n \end{pmatrix} \leftrightarrow (D_h + \tau A_\alpha) y_* = r^n, \tag{17.1}$$

$$\begin{pmatrix} \rho_h E_1 & 0 \\ -\beta\tau R & K_h^{-1} E_2 \end{pmatrix} \begin{pmatrix} v_{\bar{t}}^{n+1} \\ \sigma_{\bar{t}}^{n+1} \end{pmatrix} = \begin{pmatrix} \rho_h v_* \\ K_h^{-1} \sigma_* \end{pmatrix} \leftrightarrow (D_h + \tau A_\beta) y_{\bar{t}}^{n+1} = D_h y_* \tag{17.2}$$

and the assertion of the explicit stability of the discrete model (17) becomes obvious. Also, obvious is the assertion that the parallelization degree of (17) is the same for all $0 \le \alpha, \beta \le 1$. Observe also that $(D_h + \tau A_\alpha) = U$ in (17.1) is upper triangular, while $(D_h + \tau A_\beta) = L$ in (17.2) is lower triangular. Thus, the approximate factorization yields

$$(D_h + A_{\alpha,\beta}) = U D_h^{-1} L + O(\tau^2). \tag{18}$$

Further, we must determine the particular choice of the parameters $0 \le \alpha, \beta \le 1$ in the discrete model (16).

Theorem 2. *For $\alpha = \beta = 0.5$, the discrete model* (16) *"inherits" the conservation law of the total mechanical energy* (12) *(see* [6]*):*

$$(J_h)_{\bar{t}}^{n+1} = (\rho_h f_h, v_{0.5})_{H_h^1}, \tag{19}$$

where $(J_h)^n = \big(J_{1h}^n\big) + (J_{2h})^n = 0.5\big(\rho_h v_h^n, v_h^n\big)_{H_h^1} + 0.5\big(\varepsilon_h^n, \sigma_h^n\big)_{H_h^2}$.

In proving Theorem 2, we use the obvious consequence of (16):

$$\big(\rho_h v_{\bar{t}}^{n+1}, v_\beta\big)_{H_h^1} + \big(K_h^{-1} \sigma_{\bar{t}}^{n+1}, \sigma_\alpha\big)_{H_h^2} = (\rho_h f_h, v_\beta)_{H_h^1},$$

which, together with

$$\sigma_\alpha = \sigma_{0.5} + \tau(\alpha - 0.5)\sigma_{\bar{t}}^{n+1}, \quad v_\beta = v_{0.5} + \tau(\beta - 0.5) v_{\bar{t}}^{n+1},$$

gives

$$\begin{gathered} (J_h)_{\bar{t}}^{n+1} + \tau Q(\alpha, \beta) = (\rho_h f_h, v_\beta)_{H_h^1}, \quad Q(\alpha, \beta) = Q(\alpha) + Q(\beta), \\ Q(\alpha) = (\alpha - 0.5)(R_h^* K_h R_h v_\beta)_{H_h^1}, \quad Q(\beta) = (\beta - 0.5)\big(\rho_h v_{\bar{t}}^{n+1}, v_{\bar{t}}^{n+1}\big)_{H_h^1}. \end{gathered} \tag{20}$$

Thus, Theorem 2 becomes a consequence of (20).

The appearance of the summand $\tau Q(\alpha, \beta)$ in the conservation law (20) is as a rule connected with the "approximate viscosity" of the difference scheme (16). In this event, the word "viscosity" is endowed with no physical meaning. But then the question arises: What physical process does the discrete model (16)

with conservation law (20) correspond to? For $\alpha = \beta = 0.5$, the answer to this question is given in Theorem 2. On the other hand, if $\alpha \geq 0.5$ and $\beta \geq 0.5$ then $Q(\alpha, \beta) \geq 0$ and, for $f_h = 0$, (20) implies

$$J_h^{n+1} \leq J_h^n \leq \cdots \leq J_h^1 \leq J_h^0, \tag{21}$$

which means uniform stability with respect to the initial data in the energy norm J_h with all implications in [7, Chapters II and III] about to the convergence of the difference scheme (16). But (21) presumes the presence of a dissipation mechanism, which is absent in the continuous basic model (11).

In this connection, let us focus on the sufficiently substantial case of (16) ($\alpha \geq 0.5$ and $\beta = 0.5$) for which

$$(J_h)_{\bar{t}}^{n+1} + \tau(\alpha - 0.5)(R_h^* K_h R_h v_{0.5}, v_{0.5})_{H_h^1} = (\rho_h f_h, v_{0.5})_{H_h^1}, \tag{22}$$

and consider a dynamical problem for a Kelvin (Kelvin–Voigt) viscoelastic isotropic medium. In contrast to (1), the state equation is written down in this case as

$$\sigma_1' = K_1 \varepsilon_1 + \eta_1 \frac{\partial \varepsilon_1}{\partial t}, \quad \sigma_2' = K_2 \varepsilon_2 + \eta_2 \frac{\partial \varepsilon_2}{\partial t}. \tag{23}$$

The scalar functions $\eta_1(M) > 0$ and $\eta_2(M) > 0$ of (23) define the volume and shift viscosities of the medium under consideration. For the sake of some simplifications, put $\eta_1 = \eta_2 = \eta$. Then

$$\sigma' = K\varepsilon + \eta \frac{\partial \varepsilon}{\partial t} = \sigma + \eta \frac{\partial \varepsilon}{\partial t} = \sigma + \eta R v.$$

Thus, for the dynamical problem of an isotropic viscoelastic medium of "Kelvin type," we arrive at the continuous basic model

$$\rho \frac{\partial v}{\partial t} + R^* \sigma + R^* \eta R v = \rho f, \quad K^{-1} \frac{\partial \sigma}{\partial t} - R v = 0 \tag{24}$$

with the conservation law

$$\frac{\partial J}{\partial t} + (R^* \eta R v, v)_{H^1} = (\rho f, v)_{H^1}. \tag{25}$$

The physical processes described by continuous basic models (11) and (24) are fundamentally different from one another. In the first case, the process is equilibrium (reversible); and, in the second case, the process is irreversible. The measure of the irreversibility of the process is the entropy $s(t)$ relative to the given volume V. For the isotropic viscoelastic model (24) in [8, Part 1, § 16],

$$\rho T \frac{ds}{dt} = \Phi, \quad \Phi = (R^* \eta R v, v)_{H^1}, \tag{26}$$

where T is the absolute temperature, Φ is the dissipative function characterizing the change of the mechanical energy $J(t)$ at the passage $t \to 6t + \Delta t$. With account taken of (26), the conservation law (25) is rewritten as

$$\frac{\partial J}{\partial t} + \rho T \frac{ds}{dt} = (\rho f, v)_{H^1} \leftrightarrow \frac{\partial J}{\partial t} + \Phi = (\rho f, v)_{H^1}, \tag{27}$$

from which for $f = 0$ it follows that (cf. (21)!)

$$J(t + \tau) \leq J(t). \tag{28}$$

Finally, we point out that the conservation law for the discrete elastic model (16) for $\alpha \geq 0.5$ and $\beta = 0.5$ can be written down as

$$\begin{gathered} (J_h)_{\bar{t}}^{n+1} + \Phi_h = (\rho_h f_h, v_{0.5})_{H_h^1}, \quad \Phi_h = (R_h^*(\eta_*)_h R_h v_{0.5}, v_{0.5})_{H_h^1}, \\ (\eta_*)_h = \tau(\alpha - 0.5) K_h. \end{gathered} \tag{29}$$

Together with (27), this leads to the following result:

Theorem 3. *For $\alpha > 0.5$ and $\beta = 0.5$, the discrete model of the dynamics of the elastic medium* (16) *and the discrete model of the dynamics of an isotropic viscoelastic medium of "Kelvin type" whose viscosity $(\eta_*)_h$ is defined in* (29) *are equivalent.*

The factorization of the operator B_h in (16) leads to an explicitly solvable nonequilibrium model (17). Therefore, the disbalance of the total mechanical energy is inevitable, and, having computed ($t = t_* = k\tau$), we instead of $J_h^k = J_h^0$ will have

$$J_h^k + 0.125\tau^2\big(L_h v_h^0, v_h^0\big)_{H_h^1} = J_h^0 + 0.125\tau^2(L_h v_h^k, v_h^k), \quad L_h = R_h^* K_h R_h. \tag{30}$$

Consider the spectral problem

$$L_h q_m = \nu_m \rho_h q_m, \quad q_m \in H_h^1, \quad 0 < \nu_1 < \nu_2 < \cdots < \nu_{N-1} < \nu_N = \nu_{\max}. \tag{31}$$

Some lower bound for $\nu_{\max}$ is known: $h^2\nu_{\max} \le c^2(V_h)$, and the domain constant $c^2(M_h)$ does not depend on h. If $J_h^k ¡ J_h^0$ in (30), then from (30) and (31) it follows that

$$\big(1 - J_h^k/J_h^0\big) \le 0.25\omega_1 = \varepsilon_1. \tag{32}$$

Here and below, $\omega = c\tau/h$ is the Courant number. If $J_h^0 < J_h^k$ in (30) then, instead of (32), we have

$$\big(1 - J_h^0/J_h^k\big) \le 0.25\omega_1 = \varepsilon_1. \tag{33}$$

Estimates (32) and (33) make it possible, using the parameter ω_1, to "control" the disbalance of the total mechanical energy because ε_1 is the relative measure of disbalance in (32) and (33).

Estimates (32) and (33) hold also for the explicitly solvable discrete models ($17; 0.5 < \alpha \le 1, \beta = 0.5$). Here ε_1 of (32) and (33) must be replaced by $\varepsilon_2 = 0.5\alpha^2\omega_2^2$. If $\omega_1 = \omega_2$ then $\varepsilon_2 > \varepsilon_1$, and the relative measure of the disbalance of the total mechanical energy for ($17; 0.5 < \alpha \le 1, \beta = 0.5$) is surely greater than for ($17; \alpha = \beta = 0.5$).

Here we indicate another way [9, Chapter III] for the basic problem (15) of constructing economical nonequilibrium models with controlled disbalance of the total mechanical energy. As above, for simplifying the notation, we assume that $f_h = 0$. Thus, suppose that $w_h^n = y^n = (v^n, \sigma^n)^T$ is given. Carry out the passage $y^n \to y^{n+1}$ by means of the economical model

$$\varepsilon_{\bar t}^{n+1} - R_h v_h^n = 0 \to \sigma_h^{n+1} = K_h \varepsilon_h^{n+1} = 0 \to \rho v_{\bar t}^{n+1} + R_h^* \sigma_h^{n+1} = 0, \tag{34}$$

for which

$$J_h^{n+1} + 0.5\tau^2\big(L_h v_h^n, v_h^n\big)_{H_h^1} = J_h^n + 0.5\tau^2\big(\Lambda_h \sigma_h^{n+1}, \sigma_h^{n+1}\big)_{H_h^2}, \quad \Lambda_h = R_h 1/\rho R_h^*. \tag{35}$$

For the passage $y^{n+1} \to y^{n+2}$, use the economical model

$$\rho v_{\bar t}^{n+2} + R_h^* \sigma_h^{n+1} = 0 \to \varepsilon_{\bar t}^{n+2} - R_h v_h^{n+2} = 0, \tag{36}$$

for which

$$J_h^{n+2} + 0.5\tau^2\big(\Lambda_h \sigma_h^{n+1}, \sigma_h^{n+1}\big)_{H_h^2} = J_h^{n+1} + 0.5\tau^2\big(L_h v_h^{n+2}, v_h^{n+2}\big)_{H_h^1}. \tag{37}$$

As a consequence of (35) and (37), we have

$$J_h^{n+2} + 0.5\tau^2\big(L_h v_h^n, v_h^n\big)_{H_h^1} = J_h^n + 0.5\tau^2\big(L_h v_h^{n+2}, v_h^{n+2}\big)_{H_h^1} \tag{38}$$

and so, having finished the computation for k even, we obtain

$$J_h^k + 0.5\tau^2\big(L_h v_h^0, v_h^0\big)_{H_h^1} = J_h^0 + 0.5\tau^2\big(L_h v_h^k, v_h^k\big)_{H_h^1}. \tag{39}$$

It follows from (39) that (32) and (33) hold also for the economical nonequilibrium model (34), (36), where we must take $\varepsilon_3 = \omega_3^2$ as ε_1 now. The choice $\omega_i = c\tau_i/h$ also fixes the measure of the disbalance of the total mechanical energy ε_i, $i = 1, 2, 3$:

$$\varepsilon_1 = 0.25\omega_1^2, \quad \varepsilon_2 = 0.5\alpha\omega_2^2, \quad \varepsilon_3 = \omega_3^2. \tag{40}$$

Consequently,

$$\text{if } \varepsilon_3 = \varepsilon_2 = \varepsilon_1 \quad \text{then } \omega_1 > \omega_2 > \omega_3. \tag{41}$$

Therefore, if t_* is the execution time of the computation then

$$ht_*/c = k_1\omega_1 = k_2\omega_2 = k_3\omega_3, \tag{42}$$

which, together with (41) gives $k_1 < k_2 < k_3$.

The increase of k_i increases the number of passages $y^n \to y^{n+1}$, and so the total "computation cost" at the passage $y^0 \to y^{k_i}$. From this standpoint, the discrete nonequilibrium model $(17; \alpha = \beta = 0.5)$ is more preferable than $(17; 0.5\alpha \le 1, \beta = 0.5)$ and (34), (36). Thus, some sufficient grounds appear for adopting the nonequilibrium discrete model $(17; \alpha = \beta = 0.5)$ as basic in the numerical realization of the continuous basic model (15).

The above means also that the applied software (the application suite for specific problems of the dynamics of an elastic body) should be constructed on the basis of the explicitly solvable discrete model $(17; \alpha = \beta = 0.5)$ in its standard realization (17.1), (17.2). It is especially worth noting that it is the property of explicit solvability of the algorithms under consideration that makes it possible to use multiprocessor computing complexes of various architectures as basic computers.

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A. N. Konovalov
Institute of Computational Mathematics and Mathematical Geophysics
Novosibirsk State University, Novosibirsk, Russia
E-mail address: kan@sscc.ru

Yu. P. Popov
Keldysh Institute of Applied Mathematics, Moscow, Russia
E-mail address: popov@keldysh.ru