Efficient Solution Methods for Inverse Problems with Application to Tomography
Inverse Problems and Regularization

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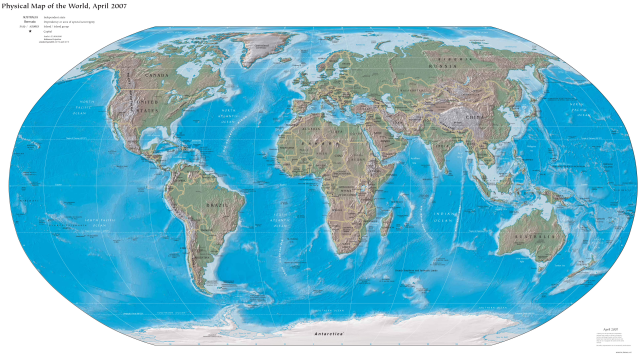
Novosibirsk NSU, October, 2011

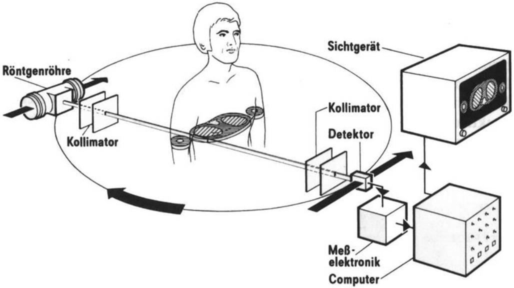
World


Physical Map of the World, April 2007
CANADA
UNITED STATES
NORTH PACIFIC OCEAN
NORTH ATLANTIC OCEAN
RUSSIA
CHINA
BRAZIL
SOUTH PACIFIC OCEAN
SOUTH ATLANTIC OCEAN
INDIAN OCEAN
AUSTRALIA
Antarctica
April 2007

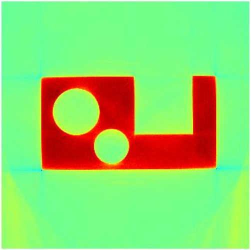
Germany

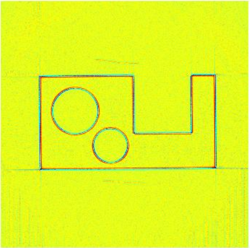

BUNDESREPUBLIK DEUTSCHLAND
DÄNEMARK
Nordsee
Ostsee
POLEN
NIEDER-
LANDE
BEL-
GIEN
LU-
XEM-
BURG
FRANKREICH
SCHWEIZ
ÖSTERREICH
TSCHECHIEN
Niedersachsen
Schleswig-
Holstein
Mecklenburg-
Vorpommern
Brandenburg
Sachsen-
Anhalt
Sachsen
Thüringen
Hessen
Nordrhein-
Westfalen
Rheinland-Pfalz
Saarland
Bayern
Baden-
Württemberg
Berlin
Hamburg
Bremen
Hannover
Frankfurt
Stuttgart
München
Nürnberg
Dresden
Leipzig

Application in Tomography



Dimensioning from Real Data





Cone-Beam Data provided by Maisl, Schorr, 2011 Fraunhofer IZFP

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Movie from Siemens, 2008

Content

Content

Inverse Problems

Let

$$A : \mathcal{X} \to \mathcal{Y}$$

where $\mathcal{X}$, $\mathcal{Y}$ Banach- or Hilbert spaces
A (linear), continuous (compact)
and injective (in Banach space case). Solve for given $g \in Y$

$$Af = g$$

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$$Af = g$$

$$(\Delta + f)u = g$$

Content

Inverse Problems

OBSERVATION $g \rightsquigarrow$ SEARCHED-FOR DISTRIBUTION f

$$Af = g$$

$A \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ Integral - or Differential - Operator

Difficulties:

- Solution does not exist
- Solution is not unique
- Solution does not depend continuously on g

$\Rightarrow$ Problem ill - posed (mal posé) (Hadamard, 1923)

Pseudo - Inverse in Hilbert Spaces

Remedy: Define

$$A^+ g := \arg \min_{f \in \mathcal{N}(A)^\perp} \|Af - g\|_Y \iff A^* A A^+ g = A^* g$$

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Properties:

- A^+ continuous $\iff \mathcal{R}(A) = \overline{\mathcal{R}(A)}$
- $(v_\mu, u_\mu; \sigma_\mu)_\mu \subset \mathcal{X} \times \mathcal{Y} \times \mathbb{R}_0^+$ singular system:

$$A^+ g = \sum_{\sigma_\mu > 0} \sigma_\mu^{-1} \langle g, u_\mu \rangle_{\mathcal{Y}} v_\mu$$

Degree of Ill-Posedness

Worst Case Error:

$$e_\nu(\varepsilon, \rho) = \sup\{\|f\| : f \in \mathcal{N}(A)^\perp, \|Af\| \leq \varepsilon, \|f\|_\nu \leq \rho\}$$

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Which Norms ?

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$$\|Af\|_{L_p} \simeq \|f\|_{W^{\alpha,p}}$$

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- Spaces defined via SVD

$$\|f\|_\nu = \left(\sum_n \sigma_n^{-2\nu} | < f, v_n > |^2\right)^{1/2}$$

Loss of Accuracy

Assumptions:

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Ill-Posedness of Problem

Regularization Methods

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if

$$\lim_{\varepsilon \to 0} \sup\{\gamma(\varepsilon, g^\varepsilon) : g^\varepsilon \in \mathcal{Y}, \quad \|g - g^\varepsilon\|_{\mathcal{Y}} < \varepsilon\} = 0\,,$$

such that

$$\lim_{\varepsilon \to 0} \sup\{\|R_{\gamma(\varepsilon, g^\varepsilon)} g^\varepsilon - A^+ g\|_{\mathcal{X}} : g^\varepsilon \in \mathcal{Y},\ \|g - g^\varepsilon\|_{\mathcal{Y}} < \varepsilon\} = 0$$

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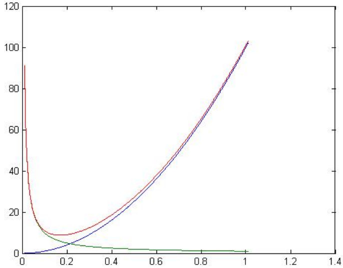
$\gamma = \gamma(\varepsilon, g^\varepsilon)$: a posteriori - parameter choice

$\gamma = \gamma(\varepsilon)$: a priori - parameter choice

Typical Behaviour of Reconstruction Error for Inexact Data

$$\|f_\gamma^\varepsilon - f\| \leq \|f_\gamma^\varepsilon - f_\gamma\| + \|f_\gamma - f\|$$



Order Optimality

Method order optimal if there exists regularizaiton parameter $\gamma = \gamma(\varepsilon, \rho)$ such that

$$\|R_\gamma g^\varepsilon - A^+ g\| \leq c\varepsilon^{\nu/(\nu+1)}\rho^{1/(\nu+1)}$$

Generation of Regularization Methods

At least two main approaches:

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  like Tikhonov-Phillips, Landweber etc.

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- Variational Methods
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- Regularization by linear functionals
  like Likht, Backus-Gilbert, L.-Maaß

Strategy

Information in the equation is not sufficient to determine a solution

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  Pucci, 55, Tikhonov, 63, Phillips, 62, Levenberg, 44, Marquardt 63
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  Likht, 67, Backus-Gilbert, 67, Shepp-Logan, 74, Smith,80,
  Grünbaum, 80, L., 89, L.-Maass, 90

Truncated Singular Value Decomposition

$$R_\gamma g = \sum_\mu \sigma_\mu^{-1} \langle g, u_\mu \rangle_{\mathcal{Y}} v_\mu$$

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Miller, 70

Additional Information

Known: $f \in V \subset X$

$$\min_{f \in V} \|Af - g\|$$

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Example:

- $V = \{f \geq 0\}$
  Convex optimization problem
- $V = \{f : \Omega(f) \leq \rho\}$
  Use Lagrangian multiplier

$$\min_{f} \left(\|Af - g\|^2 + \gamma \Omega(f) \right)$$

- Generalized Tikhonov - Phillips - Regularization

If $\Omega(f) = \|Bf\|^2$ and B linear:

$$(A^*A + \gamma B^*B)f = A^*g$$

Tikhonov, 63, Phillips, 62
For $B = I$:

$$F_\gamma(t) = t^2/(t^2 + \gamma^2)$$

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Maximum Entropy Methods (nonlinear method)

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Maximum Entropy Methods (nonlinear method)

$$\Omega(f) = \|f\|^2_{\ell_0}$$

Sparse Reconstruction
Computationally feasible: $\Omega(f) = \|f\|^2_{\ell_1}$

Iterative Methods

- Landweber
- Kaczmarz
- Conjugate Gradient
- Marquardt - Levenberg

Regularization parameter : Stopping rule

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Regularization parameter : Stopping rule
Too much work is dangerous !!
Relation between Kaczmarz and SOR : Björck - Elving, 1979

Projection Methods

Regularization parameter : step size
Natterer, 78

Selection of regularization parameter

- trial and error
- $1/$ signal -to - noise ratio
- (Generalized) cross validation
- Discrepancy Prinziple: $\|AT_\gamma g^\varepsilon - g^\varepsilon\| \simeq \varepsilon$
- L - curve (Hansen)

Deterministic vs. Stochastic ?

$$\left(A^*CA + \gamma^2 B^* B\right) f_\gamma = A^* C g$$

$$\Leftrightarrow$$

$$\min_f \left\{ \|Af - g\|_C^2 + \gamma^2 \|f\|_B^2 \right\}$$

$C = R_{\xi\xi}$	Covariance Op. for f	C	weight resp. data error
$B = R_{\zeta\zeta}$	Covariance Op. for g	B	weight for additional info.

Best linear estimator
Bayes estimate

Tikhonov-Phillips
(Marquardt-Levenberg)

Nonlinear Regularization Methods

$$\|Af - g\|^2 + \gamma^2 \Omega(f)$$

- $\Omega(f) = -\int f(x) \ln f(x) dx$ Entropy Maximization
- $\Omega(f) = \|\nabla f\|_{L_1}$ Total Variation
- Higher order total variation: Yu, Yang, Jiang, Wang, Inverse Problems, 2010

Regularization by Linear Functionals

- Likht, 1967
- Backus-Gilbert, 1967
- L.-Maass, 1990
- L, 1996

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Approximate Inverse: choose mollifier $\delta_x^\gamma \approx \delta_x$,$\delta_x^\gamma \in \mathcal{X}^*$ and solve with dual operator A^* the auxiliary problem

$$A^* \psi_x^\gamma = \delta_x^\gamma$$

Solve with precomputed reconstruction kernel $\psi_x^\gamma \in \mathcal{Y}^*$

$$f_\gamma(x) := E_\gamma f(x) := \delta_x^\gamma f = \psi_x^\gamma g$$

$S_\gamma g(x) := \psi_x^\gamma g$ is called Approximate Inverse.

Example 1:

Consider

$$A : L_2(0,1) \to L_2(0,1)$$

with

$$Af(x) = \int_0^x f(y)dy = g(x)$$

Then

$$f = g'$$

Mollifier

$$\delta_x^\gamma(y) = \frac{1}{2\gamma}\chi_{[x-\gamma,x+\gamma]}(y)$$

Example 1 continued

Auxiliary Equation:

$$A^*\psi_x^\gamma(y) = \int_y^1 \psi_x^\gamma(t)dt = \delta_x^\gamma(y)$$

leading to

$$\psi_x^\gamma(y) = \frac{1}{2\gamma}(\delta_{x+\gamma} - \delta_{x-\gamma})(y)$$

and

$$S_\gamma g(x) = \frac{g(x+\gamma) - g(x-\gamma)}{2\gamma}$$

Linear Regularization

$$
\begin{array}{ccccc}
 & & & Y_1 & \\
 & & \overset{A^+}{\swarrow} & \uparrow & \tilde{M}_\gamma \\
A: & X & \longrightarrow & Y & \\
M_\gamma & \uparrow & \overset{\overline{A^+}}{\swarrow} & & \\
 & X_{-1} & & &
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\end{array}
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$$S_\gamma = M_\gamma \overline{A^+} = A^+ \tilde{M}_\gamma$$

Realization

- Filter Methods:

$$M_\gamma f(x) = \sum_n F_\gamma(\sigma_n) < f, v_n > v_n(x)$$

$$\tilde{M}_\gamma g(x) = \sum_n F_\gamma(\sigma_n) < g, u_n > u_n(x)$$

- Linear Functionals

$$M_\gamma f(x) = \delta_x^\gamma f$$

Filter Methods are special Cases of Regularization with Linear Funcitonals

Put

$$\delta_x^\gamma(y) = \sum F_\gamma(\sigma_n) v_n(x) v_n(y)$$

then

$$R_\gamma g(x) = \sum F_\gamma(\sigma_n)\sigma_n^{-1} < g, u_n > v_n(x)$$

can be written as

$$R_\gamma g(x) = \psi_x^\gamma g$$

where

$$\psi_x^\gamma(y) = \sum F_\gamma(\sigma_n)\sigma_n^{-1} v_n(x) u_n(y)$$

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The other direction is not always possible !

Least Squares is a special Approxiamte Inverse

$X_N \subset X$ finite dimensional space, $X_N = \text{span}\{\varphi_1, \ldots, \varphi_N\}$.
Then

$$f_N = \sum_{\nu=1}^{N} \alpha_\nu \varphi_\nu$$

where

$$B\alpha = b, \; B_{\mu\nu} =< A\varphi_\mu, A\varphi_\nu >, \; b_\mu =< g, \varphi_\mu >$$

Let $\Phi = (\varphi_1, \ldots, \varphi_N)^\top$ and $C = B^{-1}$. Then the reconstruction kernel is

$$\psi_x^N(y) = (A\Phi(y)^\top C^\top \Phi(x)$$

Content

Restrict Resolution: Approximate Inverse

LOUIS, MAASS, Inverse Problems, 1990
L., Inverse Problems,1996, 1999
Compare: Likht, 67,Backus-Gilbert, 67, Grünbaum, 80, Smith 80
LOUIS, SCHUSTER, Inverse Problems, 1996

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Given:

- $A : L^2(\Omega_1, \mu_1) \to L^2(\Omega_2, \mu_2)$ linear, continuous

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Given:

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- Mollifier $\delta_x^\gamma \approx \delta_x$

$$f_\gamma(x) = \langle f, \delta_x^\gamma \rangle$$

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Idea:

- Solve

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$$f_\gamma(x) = \langle f, \delta_x^\gamma \rangle$$

Idea:

- Solve

$$A^* \psi_x^\gamma = \delta_x^\gamma$$

- Compute

$$f_\gamma(x) = \langle f, \delta_x^\gamma \rangle_{L^2(\Omega_1, \mu_1)} = \langle Af, \psi_x^\gamma \rangle_{L^2(\Omega_2, \mu_2)}$$

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Approximate Inverse in L^2-Spaces

Data g given

Approximate Inverse in L^2-Spaces

Data g given

Approximate Inverse

$$S_\gamma g(x) := \langle g, \psi_x^\gamma \rangle_{L^2(\Omega_2, \mu_2)}$$

Approximate Inverse in L^2-Spaces

Lemma

If $g \in \mathcal{D}(A^+)$, $\psi_x^\gamma \in \mathcal{N}(A^*)^\perp$, *then*

$$\lim_{\gamma \to 0} S_\gamma g = A^+ g$$

Approximate Inverse in L^2-Spaces

Lemma

If $g \in \mathcal{D}(A^+)$, $\psi_x^\gamma \in \mathcal{N}(A^*)^\perp$, *then*

$$\lim_{\gamma \to 0} S_\gamma g = A^+ g$$

The approximate Inverse is a

Regularization Method

Invariances

Theorem (L, 1997)

Let the operators T_1, T_2 intertwine with A^*; i.e.,

$$T_1 A^* = A^* T_2$$

and solve for a reference mollifier E^γ the equation

$$A^* \Psi^\gamma = E^\gamma$$

Then the general reconstruction kernel for the general mollifier $\delta^\gamma = T_1 E\gamma$ is

$$\psi^\gamma = T_2 \Psi^\gamma$$

Example: Convolution Equation

Consider $A : L_2(R^N) \to L_2(R^N)$ as

$$Af(x) = \int_{R^N} k(x-y)f(y)dy$$

Let $T_1^x = T_2^x = T^x$ be the translation $T^x f(y) = f(y-x)$. Then

$$T^x A^* = A^* T^x$$

Consequence: Solve for the fixed mollifier $E_\gamma(y)$ the equation $A^*\Psi_\gamma = E_\gamma$ and put

$$\psi_x^\gamma(y) = T^x \Psi^\gamma(y) = \Psi^\gamma(y-x)$$

Example: Tomography

Mollifier

$$\delta_x^\gamma(y) = E_\gamma(\|x - y\|)$$

Then the reconstruction kernel is

$$\psi_x^\gamma(\omega, s) = \Psi_\gamma(s - x^\top \omega)$$