Efficient Solution Methods for Inverse Problems with Application to Tomography
Approximate Inverse

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Content

1. Importance of Mathematics
2. Approximate Inverse - Theorie
3. Approximate Inverse - Examples
4. Feature Reconstruction
5. Tomography and Edge Detection
6. Radon Transform and Diffusion
7. Radon Transform and Wavelet Decomposition
8. Nonlinear Problems
9. References

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Content

1. Importance of Mathematics
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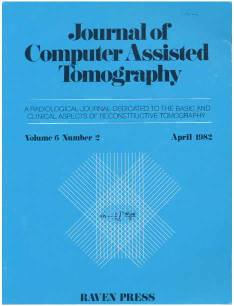
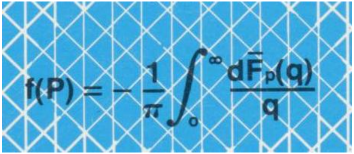
Importance of Mathematics





Content

1 Importance of Mathematics
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Regularization by Linear Functionals

- Likht, 1967
- Backus-Gilbert, 1967
- L.-Maass, 1990
- L, 1996

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Approximate Inverse: choose mollifier $\delta_x^\gamma \approx \delta_x$,$\delta_x^\gamma \in \mathcal{X}^*$ and solve with dual operator A^* the auxiliary problem

$$A^* \psi_x^\gamma = \delta_x^\gamma$$

Solve with precomputed reconstruction kernel $\psi_x^\gamma \in \mathcal{Y}^*$

$$f_\gamma(x) := E_\gamma f(x) := \delta_x^\gamma f = \psi_x^\gamma g$$

$S_\gamma g(x) := \psi_x^\gamma g$ is called Approximate Inverse.

Example 1:

Consider

$$A : L_2(0,1) \to L_2(0,1)$$

with

$$Af(x) = \int_0^x f(y)dy = g(x)$$

Then

$$f = g'$$

Mollifier

$$\delta_x^\gamma(y) = \frac{1}{2\gamma}\chi_{[x-\gamma,x+\gamma]}(y)$$

Example 1 continued

Auxiliary Equation:

$$A^*\psi_x^\gamma(y) = \int_y^1 \psi_x^\gamma(t)dt = \delta_x^\gamma(y)$$

leading to

$$\psi_x^\gamma(y) = \frac{1}{2\gamma}(\delta_{x+\gamma} - \delta_{x-\gamma})(y)$$

and

$$S_\gamma g(x) = \frac{g(x+\gamma) - g(x-\gamma)}{2\gamma}$$

Restrict Resolution: Approximate Inverse

LOUIS, MAASS, Inverse Problems, 1990
L., Inverse Problems,1996, 1999
Compare: Likht, 67,Backus-Gilbert, 67, Grünbaum, 80, Smith 80
LOUIS, SCHUSTER, Inverse Problems, 1996

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Given:

- $A : L^2(\Omega_1, \mu_1) \to L^2(\Omega_2, \mu_2)$ linear, continuous

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Given:

- $A : L^2(\Omega_1, \mu_1) \to L^2(\Omega_2, \mu_2)$ linear, continuous
- Mollifier $\delta_x^\gamma \approx \delta_x$

$$f_\gamma(x) = \langle f, \delta_x^\gamma \rangle$$

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Idea:

- Solve

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Idea:

- Solve

$$A^* \psi_x^\gamma = \delta_x^\gamma$$

- Compute

$$f_\gamma(x) = \langle f, \delta_x^\gamma \rangle_{L^2(\Omega_1,\mu_1)} = \langle Af, \psi_x^\gamma \rangle_{L^2(\Omega_2,\mu_2)}$$

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Approximate Inverse in L^2-Spaces

Data g given

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Data g given

Approximate Inverse

$$S_\gamma g(x) := \langle g, \psi_x^\gamma \rangle_{L^2(\Omega_2,\mu_2)}$$

Approximate Inverse in L^2-Spaces

Lemma

If $g \in \mathcal{D}(A^+)$, $\psi_x^\gamma \in \mathcal{N}(A^*)^\perp$, *then*

$$\lim_{\gamma \to 0} S_\gamma g = A^+ g$$

Approximate Inverse in L^2-Spaces

Lemma

If $g \in \mathcal{D}(A^+)$, $\psi_x^\gamma \in \mathcal{N}(A^*)^\perp$, *then*

$$\lim_{\gamma \to 0} S_\gamma g = A^+ g$$

The approximate Inverse is a

Regularization Method

Invariances

Theorem (L, 1997)

Let the operators T_1, T_2 intertwine with A^*; i.e.,

$$T_1 A^* = A^* T_2$$

and solve for a reference mollifier E^γ the equation

$$A^* \Psi^\gamma = E^\gamma$$

Then the general reconstruction kernel for the general mollifier $\delta^\gamma = T_1 E \gamma$ is

$$\psi^\gamma = T_2 \Psi^\gamma$$

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Example: Convolution Equation

Consider $A : L_2(R^N) \to L_2(R^N)$ as

$$Af(x) = \int_{R^N} k(x-y)f(y)dy$$

Let $T_1^x = T_2^x = T^x$ be the translation $T^x f(y) = f(y-x)$. Then

$$T^x A^* = A^* T^x$$

Consequence: Solve for the fixed mollifier $E_\gamma(y)$ the equation $A^*\Psi_\gamma = E_\gamma$ and put

$$\psi_x^\gamma(y) = T^x \Psi^\gamma(y) = \Psi^\gamma(y-x)$$

Example: Tomography

Mollifier

$$\delta_x^\gamma(y) = E_\gamma(\|x - y\|)$$

Then the reconstruction kernel is

$$\psi_x^\gamma(\omega, s) = \Psi_\gamma(s - x^\top \omega)$$

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Known Inversion Formula

Assume for $A : X \to Y$ an inversion formula is known with

$$A^{-1} = A^* B$$

Then using

$$A^* \psi_x^\gamma = \delta_x^\gamma$$

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we conclude

$$\psi_x^\gamma = B A \delta_x^\gamma$$

Radon Transform

Inversion formula 2D

$$\mathbf{R}^{-1} = c\mathbf{R}^*\mathbf{I}^{-1}$$

Hence

$$\psi^\delta = c\mathbf{I}^{-1}\mathbf{R}\delta^\gamma$$

Gaussian

$$\delta^\gamma(y) = \frac{1}{2\pi^2\gamma^2} \exp\Big(-|y|^2/(2\gamma^2)\Big)$$

Then

$$\psi^\gamma(s) = \frac{1}{2\pi^2\gamma^2}\Big(1 - \frac{\sqrt{2}s}{\gamma} D(\frac{s}{\sqrt{2}\gamma})\Big)$$

where D is the DAWSON integral

$$D(s) = \exp(-s^2) \int_0^s \exp(t^2)dt$$

Note

$$D'(s) = 1 - 2sD(s)$$

SVD known

If $A : X \to Y$ has the SVD $\{v_n, u_n; \sigma_n\}$ then the solution of $A^* \psi_x^\gamma = \delta_x^\gamma$ can be represented as

$$\psi_x^\gamma = \sum_n \sigma_n^{-1} \langle \delta_x^\gamma, v_n \rangle u_n$$

Alternative Approach

Let inversion formula again be

$$A^{-1} = A^* B$$

Let $\tilde{\delta}_y^\gamma$ be an approximation of δ_y on Y (data). Compute

$$B_\gamma = B \tilde{\delta}_y^\gamma$$

and put

$$f_\gamma = A^* B_\gamma g$$

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Features

- Edges
- Blobs
- Ridges
- Optical Flow
- Template Matching

- Wavelet Coefficients
- Diffusion for Smoothing

Typical Procedure

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- Image Reconstruction: Solve $Af = g$

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- Image Enhancement: Compute Lf

Typical Procedure

- Image Reconstruction: Solve $Af = g$
- Image Enhancement: Compute Lf
- Disadvantage: The two operations are non adjusted and possibly too time-consuming

Requirements for Repeated Solution of Same Problem with Different Data

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- Combination of reconstruction and image analysis in ONE step

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- Combination of reconstruction and image analysis in ONE step
- Precompute solution operator for efficient evaluation of the same problem with different data sets
- Use properties of the operator for fast algorithms

Target

Combine the two steps in ONE algorithm

Instead of

- Solve reconstruction part $Af = g$
- Evaluate the result Lf

Compute directly

- $LA^{\dagger}g$

Examples for Contour Reconstruction

- Lambda CT 2D : Smith, Vainberg
- Lambda 3D: L-Maass, Katsevich
- Lambda with SPECT, ET: Quinto, Öktem
- Lambda with SONAR : L., Quinto
- Simple Method = Linear Sampling = Factorization Method
- Local Tomography with Wavelets: L, Maass, Rieder, Oeckl

Image Analysis

- compute enhanced image Lf
- operate on Lf

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- operate on Lf

Example for Calculation

$$L_{k\beta}f = \frac{\partial}{\partial x_k}\left(G_\beta * f\right)$$

Edge Detection

Image Analysis

- compute enhanced image Lf
- operate on Lf

Example for Calculation

$$L_{k\beta} f = \frac{\partial}{\partial x_k}\left(G_\beta * f\right)$$

Edge Detection

- AKL 96: Compute directly derivative of solution
- Schuster, 2000 : Compute divergence of vector fields
- AKL 08,11 : General Theory

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Approximate Inverse for Features

AKL, SIAM J Imaging Sciences, 2008, Inverse Problems, 2011
Consider evaluation operator

$$L : D(L) \subset X \to X$$

Find $\delta_x^\gamma(Lf)$

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Consider evaluation operator

$$L : D(L) \subset X \to X$$

Find $\delta_x^\gamma(Lf)$

Replace Auxiliary Problem

$$A^* \psi_x^\gamma = \delta_x^\gamma$$

by

$$A^* \psi_x^{L,\gamma} = L^* \delta_x^\gamma$$

Then

$$\delta_x^\gamma(Lf) = \psi_x^{L,\gamma} g$$

AKL, 2011, Inverse Problems 27, 065010

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Proof

$$
\begin{aligned}
(Lf)_\gamma(x) &= \langle \delta_x^\gamma, Lf\rangle \\
&= \langle L^*\delta_x^\gamma, f\rangle \\
&= \langle A^*\psi_x^{L,\gamma}, f\rangle \\
&= \langle \psi_x^{L,\gamma}, f\rangle
\end{aligned}
$$

Proof

$$
\begin{aligned}
(Lf)_\gamma(x) &= \langle \delta_x^\gamma, Lf\rangle \\
&= \langle L^* \delta_x^\gamma, f\rangle \\
&= \langle A^* \psi_x^{L,\gamma}, f\rangle \\
&= \langle \psi_x^{L,\gamma}, f\rangle
\end{aligned}
$$

Conditions on e_γ such that S_γ is a regularization for determining Lf are given in the above mentioned paper.

Example: $L = \frac{d}{dx}$

Consider

$$A : C(0,1) \to C(0,1)$$

with

$$Af(x) = \int_0^x f(y)dy$$

Then

$$Lf = g''$$

Mollifier

$$\delta_x^\gamma(y) = \begin{cases} \frac{y-(x-\gamma)}{\gamma} & \text{for } x-\gamma \le y \le x \\ \frac{x+\gamma-y}{\gamma} & \text{for } x \le y \le x+\gamma \end{cases}$$

Example continued

Auxiliary Equation

$$A^*\psi_x^\gamma(y) = \int_y^1 \psi_x^\gamma(t)dt = L^*\delta_x^\gamma(y) = -\frac{\partial}{\partial y}\delta_x^\gamma(y)$$

leading to

$$\psi_x^\gamma(y) = \frac{1}{\gamma^2}(\delta_{x+\gamma} - 2\delta_x + \delta_{x-\gamma})(y)$$

and

$$S_\gamma g(x) = \frac{g(x+\gamma) - 2g(x) + g(x-\gamma)}{\gamma^2}$$

Order Optimality

Consider

$$A : W^{s,p} \to W^{s+\alpha,p}$$

with

$$\|Af\|_{L_p} \leq c_A \|f\|_{W^{-\alpha,p}}$$

and

$$L : D(L) \subset L_p \to L_p$$

with

$$\|Lf\|_{L_p} \leq c_L \|f\|_{W^{\lambda,p}} \ , \ \lambda \in [-\alpha, \beta]$$

If $\|f\|_{W^{\beta,p}} \leq \rho$ then

$$\|S_\gamma^L g^\varepsilon - Lf\|_{L_p} \leq c\varepsilon^{(\beta-\lambda)/(\alpha+\beta)} \rho^{(\alpha+\lambda)/(\alpha+\beta)}$$

with γ independent of λ, where $S_\gamma g(x) = \psi_x^\gamma g$.

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Possibilities

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- $\lambda > 0$: addtional regularization needed
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  (Ex.: Wavelet decomposition)

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- $\lambda > 0$: addtional regularization needed
  (Ex.: L differentiation)
- $\lambda = 0$: only regularization of A needed
  (Ex.: Wavelet decomposition)
- $\lambda < -\alpha$: no regularization at all
  (Ex.: diffusion)

Invariances

Theorem (L, 1997/2007)

Let the operator T_1, T_2, T_3 be such that

$$\begin{aligned} L^* T_1 &= T_2 L^* \\ T_2 A^* &= A^* T_3 \end{aligned}$$

and solve for a reference mollifier E_γ the equation

$$A^* \Psi^\gamma = L^* \delta^\gamma$$

Then the general reconstruction kernel for the general mollifier $\delta_x^\gamma = T_1 \delta_x^\gamma$ is

$$\psi_x^\gamma = T_3 \Psi^\gamma$$

and

$$f_\gamma = \langle T_3 \Psi_\gamma, g \rangle$$

Possible Calculation of Reconstruction Kernel

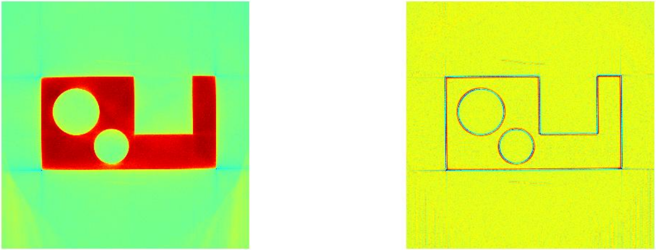
If the reconstruction of f is achieved as

$$E_\gamma f(x) = \int \psi_x^\gamma(y) g(y) dy$$

then the reconstruction kernel for calculating $L_\beta f$ can be computed as

$$\bar{\psi}_x^{\beta\gamma}(y) = L_\beta \psi_x^\gamma(y)$$

Dimensioning from Real Data



Cone-Beam Data provided by Maisl, Schorr, 2011 Fraunhofer IZFP

Content

1 Importance of Mathematics

2 Approximate Inverse - Theorie

3 Approximate Inverse - Examples

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6 Radon Transform and Diffusion

7 Radon Transform and Wavelet Decomposition

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Radon Transform

$$\mathbf{R}f(\theta, s) = \int_{\mathbb{R}^2} f(x)\delta(s - x^\top\theta)dx$$

$\theta \in S^1$ and $s \in \mathbb{R}$.

$$\widehat{\mathbf{R}f}(\theta, \sigma) = (2\pi)^{1/2}\hat{f}(\sigma\theta)$$

$$\mathbf{R}^{-1} = \frac{1}{4\pi}\mathbf{R}^*\mathbf{I}^{-1}$$

where $\mathbf{R}^*$ is the adjoint operator from L_2 to L_2 known as backprojection

$$\mathbf{R}^*g(x) = \int_{S^1} g(\theta, x^\top\theta)d\theta$$

and the Riesz potential $\mathbf{I}^{-1}$ is defined with the Fourier transform

$$\widehat{\mathbf{I}^{-1}g}(\theta, \sigma) = |\sigma|\hat{g}(\theta, \sigma)$$

Radon Transform and Derivatives

The Radon transform of a derivative is

$$\mathbf{R}\frac{\partial}{\partial x_k}f(\theta, s) = \theta_k \frac{\partial}{\partial s}\mathbf{R}f(\theta, s)$$

Invariances

Let for $x \in \mathbb{R}^2$ the shift operators $T_2^x f(y) = f(y - x)$ and $T_3^{x^\top \theta} g(\theta, s) = g(\theta, s - x^\top \theta)$ then

$$\mathbf{R}_\theta T_2^x = T_3^{x^\top \theta} \mathbf{R}_\theta$$

Let U be a unitary 2×2 matrix and $D_2^U f(y) = f(Uy)$. then

$$\mathbf{R} D_2^U = D_3^U \mathbf{R}$$

where $D_3^U g(\theta, s) = g(U\theta, s)$.

$$(T\mathbf{R})^* = \mathbf{R}^* T^*$$

Reconstruction Kernel

Theorem

Let the mollifier be given as $\delta_x^\gamma(y) = E_\gamma(\|x - y\|)$*.*
Then the reconstruction kernel for $L_k = \partial/\partial x_k$ *is*

$$\psi_x^{k,\gamma} = \theta_k \Psi_\gamma\left(s - x^\top\theta\right)$$

with

$$\Psi_\gamma = -\frac{1}{4\pi}\frac{\partial}{\partial s}\mathbf{I}^{-1}\mathbf{R}E_\gamma$$

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L not rotational invariant, hence the kernel depends on θ.

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L not rotational invariant, hence the kernel depends on θ.
Same costs for calculating $\frac{\partial f}{\partial x_k}$ as for calculating f.

Example

Let $E_{\beta\gamma}$ be given as

$$\widehat{E_{\beta}\gamma}(\xi) = (2\pi)^{-1} \operatorname{sinc}\frac{\|\xi\|\pi}{\beta} \underbrace{\operatorname{sinc}\frac{\|\xi\|\pi}{2\gamma}\chi_{[-\gamma,\gamma]}(\|\xi\|)}_{\text{Shepp-Logan kernel}}$$

Then the reconstruction kernel for $\beta = \gamma$ is

$$\psi_{\pi/h}(\ell h) = \frac{1}{\pi^2 h^3} \frac{8\ell}{\left(3 + 4\ell^2\right)^2 - 64\ell^2}, \; \ell \in Z$$

Gaussian Kernel

Kernel based on Gaussian, see Rieder, 2001

$$\psi^{\gamma}(s) = \frac{1}{2\pi}\int_0^{\infty} \sigma \exp(-\gamma\sigma^2)\cos(s\sigma)d\sigma$$

and

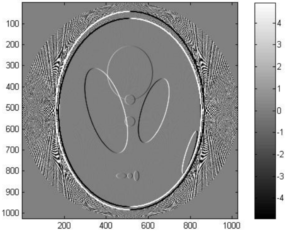
$$\psi^{k,\gamma}(s) = \theta_k \frac{d}{ds}\psi^{\gamma}(s)$$

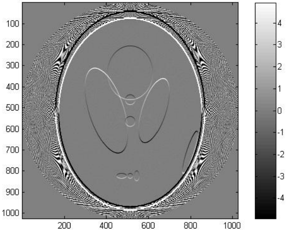
with the Dawson integral

$$D(s) = \exp(-s^2)\int_0^{s} \exp(t^2)dt$$

Formulas in Louis, Inverse Problems 27, 065010, 2011

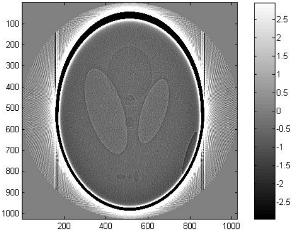
Exact Data

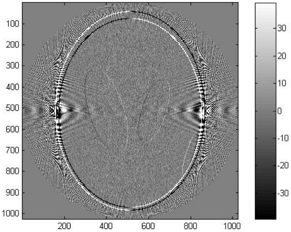




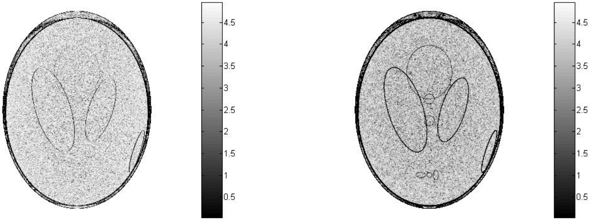
Shepp - Logan phantom with original densities

Lambda and wrong Parameter β



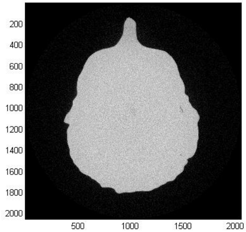


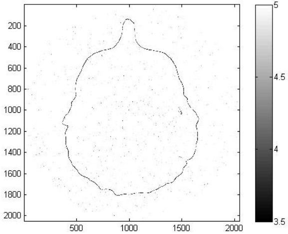
Differentiation of Image vs. New Method



Sum of absolute values of the two derivatives
Apply support theorem of Boman and Quinto.

Reconstruction from Measured Data, Fan Beam Geometry





Content

1 Importance of Mathematics
2 Approximate Inverse - Theorie
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4 Feature Reconstruction
5 Tomography and Edge Detection
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Diffusion Processes

AKL.: Inverse Problems and Imaging **4** Issue 4, 2010

Smoothing of noisy images f by diffusion

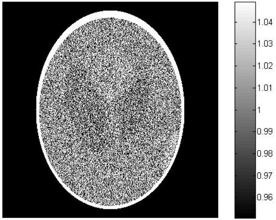
$$\begin{aligned} \frac{\partial}{\partial t} u &= \Delta u \\ u(0, x) &= f(x) \end{aligned}$$

Problem Edges are smeared out
Perona-Malik: Nonlinear Diffusion

$$u_t = \Delta \frac{u}{|\nabla u|}$$

Example

Shepp-Logan phantom, 12% noise



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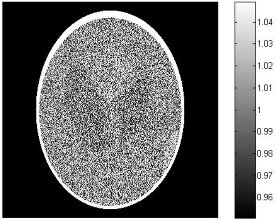
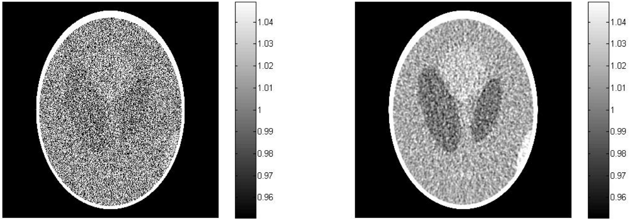


Normal reconstruction (left) and smoothing by linear diffusion, $T = 0.1$

Example

Shepp-Logan phantom, 12% noise



Normal reconstruction (left) and smoothing by linear diffusion, $T = 0.1$

Fundamental Solution

L.: Inverse Problems and Imaging, to appear

Fundamental Solution of the heat equation

$$G(t,x) = \frac{1}{(4\pi t)^{n/2}} \exp(-|x|^2/4t)$$

The solution of the heat equation for initial condition $u(0,x) = f(x)$ can be calculated as

$$L_T u(0,\cdot) = u(T,\cdot) = G(T,\cdot) * f$$

Combining Reconstruction and Diffusion

Let $\delta_x^\gamma = \delta_x$ and $L = L_T$
Then the reconstruction kernel for time $t = T$ has to fulfill

$$R^*\psi_T(x, y) = L_T^*\delta_x(y) = G(T, x - y)$$

and is of convolution type.

Reconstruction Kernel

Let

$$E_\gamma(x) = \delta_x$$

Then

$$\psi_T(s) = \frac{1}{2\pi}\int_0^\infty \sigma \exp(-T\sigma^2)\cos(s\sigma)d\sigma$$

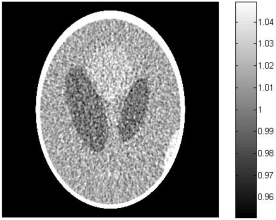
and

$$\psi_T(s) = \frac{1}{4\pi^2 T}\left(1 - \frac{s}{\sqrt{T}}D\left(\frac{s}{2\sqrt{T}}\right)\right)$$

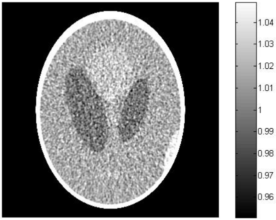
with the Dawson integral

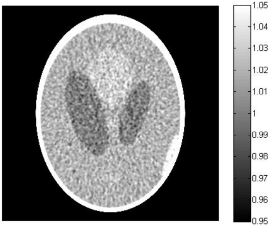
$$D(s) = \exp(-s^2)\int_0^s \exp(t^2)dt$$

Numerical Tests with noisy Data, 12 % noise



Numerical Tests with noisy Data, 12 % noise





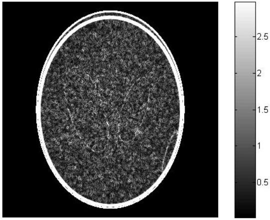
Smoothing by linear diffusion (left) and combined reconstruction (right).

Reconstruction of Derivatives for Noisy Data

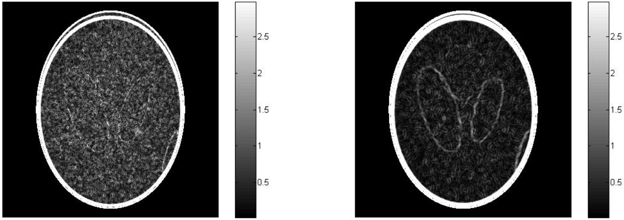
Reconstruction Kernel for Derivative in Direction x_k

$$\psi_{T,k}(s) = -\frac{\theta_k}{4\pi^2 T^{3/2}}\Big(\frac{s}{2\sqrt{T}}+(1-\frac{s^2}{2T}D(\frac{s}{2\sqrt{T}}))\Big)$$

Noise and Differentiation, 6% Noise

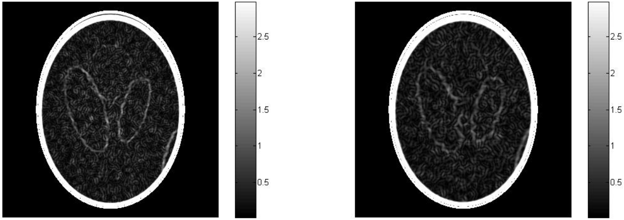


Noise and Differentiation, 6% Noise



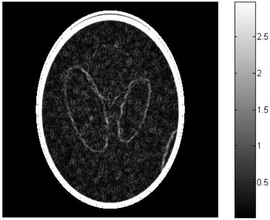
Smoothing by linear diffusion (left) and combined reconstruction (right).

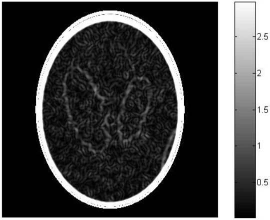
Noise and Differentiation, 6% and 12% Noise



Combined reconstruction of sum of absolute values of derivatives for 6% and 12%noise .

Noise and Differentiation, 6% and 12% Noise





Combined reconstruction of sum of absolute values of derivatives for 6% and 12%noise .

No useful results with 12% noise with separate calculation of reconstruction and smoothed derivative.

Content

1 Importance of Mathematics

2 Approximate Inverse - Theorie

3 Approximate Inverse - Examples

4 Feature Reconstruction

5 Tomography and Edge Detection

6 Radon Transform and Diffusion

7 Radon Transform and Wavelet Decomposition

8 Nonlinear Problems

9 References

Wavelet Components

- Combination of reconstruction and wavelet decomposition: L, Maass, Rieder, 1994
- Image reconstruction and wavelets :
    - Holschneider, 1991
    - Berenstein, Walnut, 1996
    - Bhatia, Karl, Willsky, 1996
    - Bonnet, Peyrin, Turjman, 2002
    - Wang et al, 2004

Inversion with Wavelets

L.,MAASS,RIEDER 94

Represent the solution of $Rf = g$ as

$$f = \sum_k c_k^M \varphi_{Mk} + \sum_m \sum_\ell d_\ell^m \psi_{m\ell}$$

Inversion with Wavelets

L.,MAASS,RIEDER 94

Represent the solution of $Rf = g$ as

$$f = \sum_k c_k^M \varphi_{Mk} + \sum_m \sum_\ell d_\ell^m \psi_{m\ell}$$

Precompute v_{Mk}, $w_{m\ell}$ for $\delta_{Mk} = \varphi_{Mk}$ and $\delta_{m\ell} = \psi_{m\ell}$ as

$$R^* v_{Mk} = \varphi_{Mk}$$

$$R^* w_{m\ell} = \psi_{m\ell}$$

Inversion with Wavelets

L.,MAASS,RIEDER 94

Represent the solution of $Rf = g$ as

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$$R^* v_{Mk} = \varphi_{Mk}$$

$$R^* w_{m\ell} = \psi_{m\ell}$$

Then

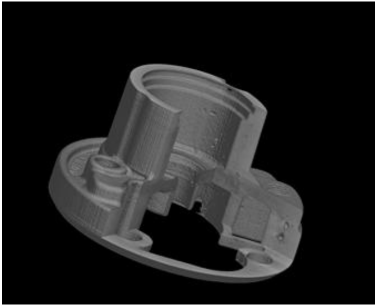
$$c_k^M = \langle g, v_{Mk} \rangle$$

$$d_\ell^m = \langle g, w_{m\ell} \rangle$$

This is joint work with Steven Oeckl
Department Process Integrated Inspection Systems
Fraunhofer IIS

New Application, first in NDT

In-line inspection in production process



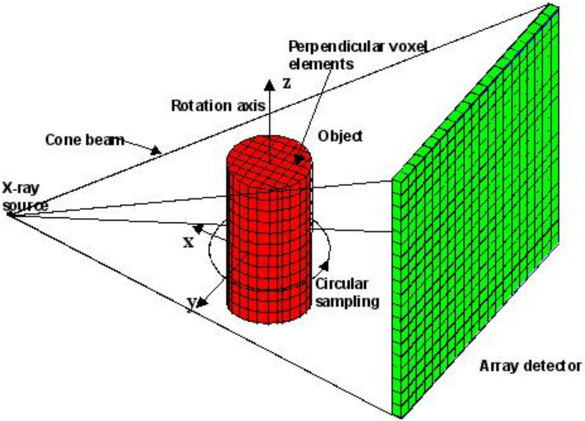
Reconstruction result:

- Three-dimensional registration of the object

Main inspection task

- Dimensional measurement
- Detection of blow holes and porosity

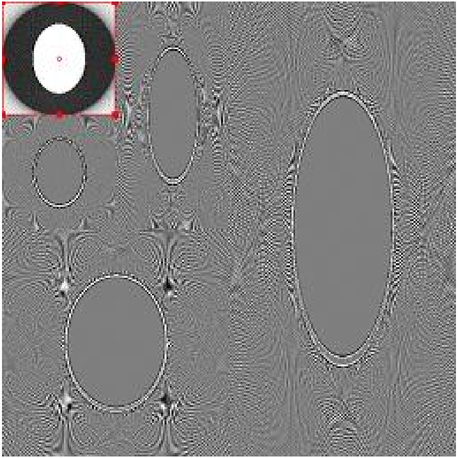
Typical Scanning Geometry in NDT



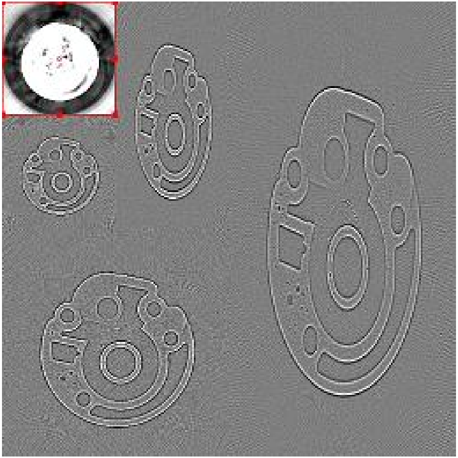
Wavelet Coefficient for 2D Parallel Geoemtry

$$
\begin{aligned}
\langle f, \psi_{jk} \rangle &= \frac{1}{4\pi} \int_{S^1} \int_R I^{-1} g(\theta, s) R_\theta \psi_{jk}(s) ds d\theta \\
&= \frac{1}{4\pi} \int_{S^1} \left(I^{-1} g(\theta, \cdot) * R_{-\theta} \psi_{j0} \right) (\langle D^j k, \theta \rangle) d\theta
\end{aligned}
$$

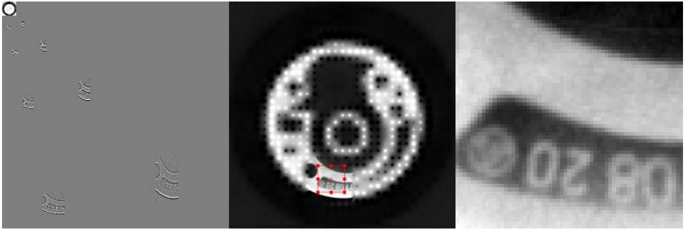
2D Shepp-Logan Phantom, Fan Beam



Cone Beam: decentralized slice



Cone Beam 'Local Reconstruction'



Content

1 Importance of Mathematics

2 Approximate Inverse - Theorie

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Considered Nonlinearity

$$Af = \sum_{\ell=1}^{\infty} A_\ell f$$

where

$$A_1 f(x) = \int k_1(x,y) f(y) dy$$

$$A_2 f(x) = \int k_2(x,y_1,y_2) f(y_1) f(y_2) dy$$

Considered by :
Snieder for Backus-Gilbert Variants, 1991
L: Approximate Inverse, 1995

Ansatz for Inversion, Presented for 2 Terms

Put

$$S_\gamma g = S_1 g + S_2 g$$

with

$$S_1 g(x) = \langle g, \psi_1(x, \cdot)\rangle$$

and

$$S_2 g(x) = \Big\langle g, \langle \Psi_2(x, \cdot, \cdot), g\rangle \Big\rangle$$

Replace g by Af and omit higher order terms. Then

$$S_\gamma A f = \underbrace{S_1 A_1 f}_{\approx L E_\gamma f} + \underbrace{S_1 A_2 f + S_2 A_1 f}_{\approx 0}$$

Determination of the Square Term

Consider

$$A : L_2(\Omega) \to \mathbb{R}^N$$

Minimizing the defect leads to the equation

$$A_1 A_1^* \Psi_\gamma(x) A_1 A_1^* = -\sum_{n=1}^{N} \psi_{\gamma,n}(x) B_n$$

where

$$B_n = \int_{\Omega\times\Omega} k_1(y_1) k_{2n}(y_1, y_2) k_1(y_2) dy_1 dy_2$$

Note: B_n is independent of x !

Example: $A : L_2 \to \mathbb{R}^N$

Vibrating String

$$u''(x) + \rho(x)\omega^2 u(x) = 0, \quad u(0) = u(1) = 0$$

$$\rho(x) = 1 + f(x)$$

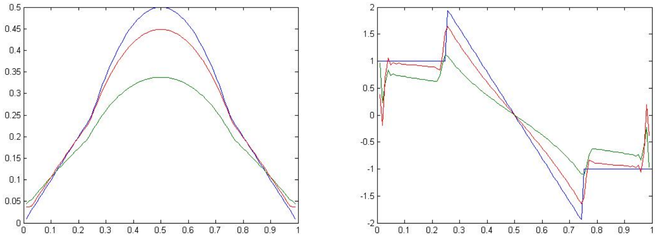
Data

$$g_n = \frac{\omega_n^2 - (\omega_n^0)^2}{(\omega_n^0)^2}, \quad , n = 1, \dots, N$$

$$k_{1,n}(y) = -2\sin^2(n\pi y)$$

$$k_{2,n}(y_1, y_2) = 4\sin^2(n\pi y_1)\sin^2(n\pi y_2)$$
$$+4\sum_{n\neq m} \frac{n^2}{n^2 - m^2} \sin(n\pi y_1)\sin(m\pi y_1)\sin(n\pi y_2)\sin(m\pi y_2)$$

Numerical Test, $L = \frac{d}{dx}$



Linear and quadratic approximation of function (left) and derivative (right).

Content

1 Importance of Mathematics

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